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Evaluating Dimensional and Distributional Contributions to Multidimensional Poverty

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Abstract: (300 words)

The adjusted headcount ratio M_0 developed by Alkire and Foster (2011a) is increasingly being applied by countries and international organizations to measure multidimensional poverty. Three properties are largely responsible for its growing use: *subgroup decomposability*, by which an assessment of subgroup contributions to overall poverty can be made, thus facilitating targeting; *dimensional breakdown*, by which an assessment of dimensional contributions to overall poverty can be made, after the poor have been identified, thereby facilitating coordination; and *ordinality*, which ensures that the method can be used in cases where variables only have ordinal meaning. Following Sen (1976), a natural question is whether sensitivity to inequality among the poor could be usefully incorporated into this methodology. We provide several answers to this motivating question. We note that there are many existing measures that are sensitive to inequality but require the use of cardinal variables. We develop a new *dimensional transfer* axiom that applies to measures using ordinal data, and search for methods related to M_0 satisfying this form of distribution sensitivity among the poor. An intuitive *attainment count transformation* is presented that converts unidimensional methods to multidimensional methods, and we show how both the original adjusted headcount ratio (used by Colombia) and an alternative version (used by Mexico) are obtained from unidimensional poverty gap measures. By replacing the poverty gap with any distribution sensitive measure we immediately obtain a multidimensional poverty method satisfying the dimensional transfer axiom. However, none of these examples satisfies the dimensional breakdown axiom. A general impossibility theorem explains why this is so: no multidimensional poverty methodology can simultaneously satisfy the dimensional transfer axiom and the three original properties. We propose alternative approaches that can capture some aspects of inequality without sacrificing the properties that make the adjusted headcount ratio so useful. The methods are illustrated with an example.

Key Words: poverty measurement, multidimensional poverty, inequality, FGT measures, decomposability, axioms, identification, ordinal variables.

JEL codes: I3, I32, D63, O1, H1

1. Introduction

There is by now much evidence that the multidimensional poverty methodology of Alkire and Foster (2011a) is well suited for real world policy applications – evaluating poverty over space and time, targeting poor populations and coordinating anti-poverty efforts among government agencies.[1] The effectiveness of the methodology originates in the properties it satisfies, including *subgroup decomposability (SD)*, by which an assessment of contribution of subgroup contributions to overall poverty can be made, thus facilitating targeting, and *dimensional breakdown (DB)*, by which an assessment of dimensional contributions to overall poverty (after the poor have been identified) can be made, thereby facilitating coordination. A third property of *ordinality (O)* allows the *adjusted headcount ratio*, or M_0 from the associated M_α class of measures, to be used in the all too common cases where variables are ordinal or even categorical, thereby ensuring its broad applicability.

A natural question to ask is whether inequality can be usefully incorporated into this form poverty measurement. Following Sen (1976), the literature on income (or unidimensional) poverty expresses the concern for inequality using a transfer principle that requires poverty to fall as result of a progressive transfer among the poor. This in turn has led to an array of distribution-sensitive income poverty measures satisfying this property.[2] The analogous discussion in the multidimensional context must confront the fact that there are two competing notions of multidimensional inequality, leading to two distinct ways of conceiving of inequality in poverty. The first, linked most closely to Kolm (1977), generalizes the notion of a progressive transfer (or more broadly a Lorenz comparison) to the multidimensional setting by applying the same bistochastic matrix to every variable.[3] This results in a coordinated "smoothing" of the distributions that preserves their means. The associated transfer principle for poverty measures requires poverty to fall, or at least not to rise, when such a smoothing is applied among the poor.

[1] See x and y (Examples that illustrate the effective use of the methods. Maybe one or two that use M1 or M2?)

[2] See Sen (1976), Clark, Hemming, and Ulph (1981), and Foster, Greer, and Thorbecke (1984) among others. It should be noted that a property depends on both the identification and aggregation steps. In unidimensional measurement, identification usually has a standard format, so we often say that the poverty measure satisfies a given property without explicitly specifying the identification method.

[3] A bistochastic matrix is a weighted average of different permutation matrices (each of which switches achievements around). When applied to an income distribution it ensures that each person's transformed income is a weighted average of all the original incomes. See Foster and Sen (1997) or Alkire et al (2014).

The second form of multidimensional inequality is linked to the work of Atkinson and Bourguignon (1982), and relies on patterns of achievements *across* dimensions. Imagine a case where one person initially has more of everything than another person, and the two persons switch achievements for a single dimension. This can be interpreted as a progressive transfer that preserves the marginal distribution of each variable, and lowers inequality by relaxing the positive association across variables. The resulting transfer principle specifies conditions under which this alternative form of progressive transfer among the poor should lower poverty, or at least not raise it.

Many multidimensional poverty methodologies satisfy one or both of these transfer principles.[4] In particular, we have shown in Alkire and Foster (2011a) that the multidimensional measures M_α satisfy the first type of transfer principle for $\alpha \geq 1$ and the second type for $\alpha \geq 0$. Note, though, that the transfer properties in the multidimensional poverty literature are "weak" in that they allow poverty to remain unchanged in the face of a progressive transfer. In particular, the adjusted poverty gap measure M_1, which is thoroughly insensitive to either form of transfer, satisfies both. It is possible to define associated strict versions of the properties that require poverty to fall as a result of a suitably strict progressive transfer, and to show that for $\alpha > 1$ the measure M_α satisfies a strict version of the first transfer principle, while for $\alpha > 0$ it can be easily transformed into a *new* measure satisfying a strict version of the second (as outlined in the paper).[5] However, each of these measures violates property *O*, thus limiting its applicability in an important way. This leads to the following natural questions: Is it possible to formulate a strict version of distribution sensitivity – by which greater inequality among the poor strictly raises poverty – that is applicable to poverty measures that use ordinal data? And can we find measures satisfying this requirement and properties *SD* and *DB*, which have proved to be so useful in practice?

This paper considers the possibility of constructing multidimensional poverty measures satisfying properties *O*, *SD*, and *DB* that also satisfy a strict form of distribution sensitivity called *dimensional transfer (DT)*. This property follows the Atkinson-Bourguignon form of

[4] See, for example, Chakravarty, Mukherjee and Renade (1998), Tsui (2002), Bourguignon and Chakravarty (2003), Chakravarty and D'Ambrosio (2006), Maasoumi and Lugo (2008), and Bossert, Chakravarty, and D'Ambrosio (2013).

[5] See Alkire and Foster (2011a) p. 485, where it is noted that one could replace the individual poverty function $M_\alpha(y_i;z)$ with $[M_\alpha(y_i;z)]^\gamma$ for some $\gamma > 0$ and average across persons.

distribution sensitivity, but with an additional proviso that poorer person is deprived in the dimension of the switch while the other person is not – so that the switch can be interpreted as the transfer of a *deprivation* to a better off person from a worse off one. Following Alkire and Foster (2011a) we show how to transform the adjusted headcount ratio M_0 to obtain measures satisfying *DT*. We then generalize to obtain an intuitive procedure for constructing multidimensional measures from unidimensional measures by constructing an *attainment count distribution* and applying a unidimensional poverty measure.[6] This transformation identifies the commonalities among the global *Multidimensional Poverty Index* (or *MPI*), the official Columbian *MPI*, and Mexico's multidimensional methodology, namely, each can be viewed as a unidimensional poverty gap measure applied to the attainment count distribution.[7]

A general theorem shows how this construction method effectively converts properties for unidimensional poverty measures into the corresponding properties for multidimensional measures. For example, *monotonicity* of an income poverty measure ensures that the resulting multidimensional poverty measure satisfies *dimensional monotonicity* as defined in Alkire and Foster (2011a). *SD* likewise follows from the associated unidimensional property. To obtain multidimensional measures that satisfy *DT*, it turns out that any unidimensional measure satisfying the unidimensional *transfer* principle, such as the FGT measure, will do. Hence, it is straightforward to construct any number of multidimensional poverty measures that satisfying *O*, *SD* and *DT*: Any subgroup decomposable, unidimensional poverty measure satisfying the *transfer* principle will generate one. Unfortunately, it is *also* true that every one of these examples violates *DB*. We identify the reasons for the violation and prove an impossibility result that demonstrates the mutual incompatibility of the four properties. In words, the highly desirable and practical properties of *subgroup decomposability*, *dimensional breakdown* and *ordinality* together prevent a poverty measure from satisfying the *dimensional transfer* property. Given the key role played by *O*, *SD* and *DB* in the successful application of multidimensional methods, we support using methods satisfying the three, augmented by information that separately accounts for inequality. In particular, the adjusted headcount ratio, which is neutral with respect to the transfers in the *DT* property and

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[6] A person's *deprivation score* is the sum of the values of deprivations experienced by the person; the *attainment count* is the sum of the values associated with the remaining dimensions – those in which the person is not deprived. See section 4.
[7] See Alkire and Santos (2010, 2014) for a discussion of the global MPI.

satisfies the other three properties outright, can be used in conjunction with other indicators that convey information on the variability of deprivations (or attainments) among the poor. The methods are illustrated with an example.

The basic definitions and notation used in this paper are given in Section 2 while Section 3 describes the key properties for multidimensional poverty measures. Section 4 presents a series of measures satisfying the dimensional transfer property and describes a method of constructing multidimensional measures from unidimensional measures. A general theorem linking unidimensional and multidimensional properties is provided, which among other things shows how to construct multidimensional measures satisfying the dimensional transfer property. Section 5 presents the impossibility result and outlines some potential ways forward with the help of examples, while Section 6 concludes.

2. Notation and Definitions

We begin with notation and definitions needed in the subsequent discussion. Let $|v|$ denote the sum of all elements in any vector or matrix v, and $\mu(v)$ signify the mean of v, or $|v|$ divided by the total number of elements in v. Let integer $n \geq 1$ represent the number of persons, where n will be permitted to range the integers, and let $i = 1,2,\ldots,n$, denote the typical person. Well-being or poverty is reflected in the achievements from a fixed, finite number $d \geq 2$ of dimensions, where the typical dimension is $j = 1,2,\ldots,d$. Let $y = [y_{ij}]$ be an $n \times d$ matrix of achievements, belonging to the domain $Y = \{y \in R_+^{nd} : n \geq 1\}$ of matrices under consideration.[8] The typical entry in y is $y_{ij} \geq 0$. We use y_i to signify the row vector of individual i's achievements, and $y_{\cdot j}$ is the column vector that provides the distribution of dimension j's achievements across people. A *deprivation cutoff* $z_j > 0$ for dimension j is compared to achievement level y_{ij} to determine when person i is deprived in j, namely when $y_{ij} < z_j$. The row vector of dimension-specific cutoffs is denoted by z.

Poverty measurement has an identification step and an aggregation step. An *identification function* ρ: $R_+^d \times R_{++}^d \rightarrow \{0,1\}$ is used to identify whether person i is poor, where $\rho(y_i; z)$ takes the value of one if person i is poor, and the value of zero otherwise. The *identification vector*

[8] We follow Alkire and Foster (2011) in assuming that achievements are represented as nonnegative real numbers, while deprivation cutoffs are strictly positive. Other assumptions are clearly possible, but are not explicitly covered here.

associated with y is the column vector r whose i^{th} entry is $\rho(y_i; z)$, while the set of persons who are identified by ρ as being poor is denoted by $Z \subseteq \{1,\ldots, n\}$. An *index* or *measure of multidimensional poverty* M: $Y \times R^d_{++} \to \mathbb{R}$ aggregates the data into an overall level $M(y; z)$ of poverty in y given z and the identification function ρ. The resulting *methodology* for measuring multidimensional poverty is given by $\mathcal{M} = (\rho, M)$. For any given dimension j, let $Y_{\cdot j}$ be the set of all column vectors $y_{\cdot j}$ of j^{th} dimensional achievements. It will sometimes be useful to focus on y and $y_{\cdot j}$ that are consistent with a given poverty status vector r. Let Y_r denote the set containing all y that are consistent with r, and let Y_{rj} denote the set of all dimension j vectors $y_{\cdot j}$ that are derived from an achievement matrix y consistent with a given identification vector r.

Alkire and Foster (2011a,b) identify and measure poverty using a vector of deprivation values and an overall poverty cutoff. Let $w_j > 0$ denote the *weight* or *deprivation value* of j and let w be the row vector satisfying $|w| = d$. The *poverty cutoff* is denoted by k, where $0 < k \leq d$. For any person i, the *deprivation score* (or *count*) c_i is the sum of the deprivation values w_j across all dimensions in which i is deprived. The *dual cutoff identification function* ρ_k is defined by $\rho_k(y_i;z) = 1$ whenever $c_i \geq k$, and $\rho_k(y_i;z) = 0$ whenever $c_i < k$. In other words, ρ_k identifies person i as poor when the deprivation count c_i is at least k and i is not poor otherwise.[9] Let $Z(k)$ be the set of persons who are identified by ρ_k as being poor. At one extreme, when $k = d$, the function ρ_k becomes *intersection* identification, in which a person must be deprived in all dimensions to be poor. When $0 < k \leq \min_j w_j$, it becomes *union* identification, in which a person need be deprived in only one dimension to be identified as poor. Thus, while our emphasis is on the intermediate cases, ρ_k includes the two limiting identification methods.

Using deprivation cutoffs and values, we convert the matrix of achievements into a matrix focusing on deprivations. Let $g^0 = [g^0_{ij}]$ denote the deprivation matrix whose typical element is given by $g^0_{ij} = w_j$ when $y_{ij} < z_j$, and $g^0_{ij} = 0$ otherwise. In words, when person i is deprived in the j^{th} dimension the associated entry g^0_{ij} is the deprivation value w_j; otherwise it is 0. Column vector $g^0_{\cdot j}$ clearly indicates those who are deprived in dimension j, while the i^{th} row vector of g^0 is person i's *deprivation vector*, denoted g^0_i. Summing the values in g^0_i yields the

[9] As noted below it is also possible to use $c_i > k$ in the definition of the poor.

deprivation count $c_i = |g_i^{\mathbf{0}}|$ as defined above, while further dividing by the maximal value d yields the *deprivation share or score* $s_i = c_i/d$, from which the column vector s of deprivation scores is constructed. It lists the intensity (or breadth) of deprivation experienced by each person.

The poverty cutoff k can be used to create a matrix focused on the deprivations of the poor. Let $g^0(k)$ denote the censored deprivation matrix whose typical element is given by $g_{ij}^0(k) = g_{ij}^0\ \rho_k(y_i;\ z)$, which leaves the entries of the poor unchanged, while changing those of the nonpoor to zero.[10] The column vector $g_{\cdot j}^0(k)$ contains w_j for every person who is both poor and deprived in dimension j, and 0 otherwise. Person i's *censored deprivation vector* $g_i^{\mathbf{0}}(k)$ is the i^{th} row of $g^0(k)$. The censored vector of deprivation shares $s(k)$, given by $s_i(k) = |g_i^{\mathbf{0}}(\mathrm{k})|/d = \rho_k(y_i;z)s_i$ for $i = 1,\ldots,n$, differs from s in that the entries of the nonpoor are set to zero.

The first poverty measure defined in Alkire and Foster (2011a) is the adjusted headcount ratio $M_0 = M_0(y;z) = \mu(g^0(k))$, or the mean of the censored deprivation matrix. It is the total value of all deprivations experienced by the poor as a share of the maximum total value of deprivations that would be obtained if everyone were fully deprived. M_0 can be expressed as the product of two intuitive partial indices, the *headcount ratio H* and the *average intensity* of poverty, denoted A. The headcount ratio arising from dual cutoff identification is define as $H = q/n,$ where $q = \Sigma_{i=1}^n \rho_k(y_i;\ z)$ is total number of poor persons identified by ρ_k. The average intensity is given by $A = |s(k)|/q = (|g_1^0(k)|+\ldots+|g_n^0(k)|)/(qd)$. It is easy to show that

$$M_0 = (|g_1^0(k)|+\ldots+|g_n^0(k)|)/(nd) = HA \qquad (1)$$

which offers a decomposition of the measure by population. In addition M_0 can be broken down by dimension as follows:

$$M_0 = (|g_{\cdot 1}^0(k)|+\ldots+|g_{\cdot d}^0(k)|)/(nd) \qquad (2)$$

We will return to these two expressions below.

Alkire and Foster (2011a) also define other measures that require more from the data – namely that each variable is cardinal – which ensures that the normalized gaps $g_{ij} = (z_j\text{-}y_{ij})/z_j$

[10] Note that in the case of union identification, the censored and original deprivation matrices are identical.

of the poor are meaningful. In this case, the censored deprivation matrix $g^0(k)$ can be replaced by the matrix $g^\alpha(k)$ having as its typical entry $g_{ij}^\alpha(k) = g_{ij}^0(k)((z_j\text{-}y_{ij})/z_j)^\alpha$ for a given $\alpha > 0$. A class of multidimensional poverty measures can then be defined by $M_\alpha(y;z) = \mu(g^\alpha(k))$ for $\alpha \geq 0$. In particular the adjusted poverty gap M_1 is sensitive to the depth of deprivation in each dimension, while the adjusted FGT or squared gap M_2 emphasizes the largest normalized gaps, and is sensitive to a particular type of multidimensional inequality in the distribution of achievements. Since our present concern is with measures that satisfy ordinality, we focus on M_0 in what follows.

3. Properties

The properties of a poverty measure specify the patterns in the underlying data the measure should ignore, the aspects it should highlight, and the kinds of policy questions it can be used to answer. This section presents properties for multidimensional poverty measures, focusing first on the traditional properties satisfied by M_0 or, more precisely, by the methodology $\mathcal{M}_{k_0} = (\rho_k, M_0)$ since properties are, in fact, joint restrictions on identification and aggregation. Only general descriptions of these properties are provided here; precise definitions and verifications can be found in Alkire and Foster (2011a). Two additional properties of $\mathcal{M}_{k_0}$ that were previously discussed, but have not yet received a rigorous treatment, are defined: *ordinality*, which ensures that the measure can be meaningfully applied to ordinal data; and *dimensional breakdown*, which allows poverty to be broken down by dimension after identification. We conclude with a new property - *dimensional transfer* - which ensures that poverty is sensitive to one form of inequality among the poor.

The properties of multidimensional poverty methodologies can be divided into the categories of *invariance*, *dominance*, and *subgroup* properties. Invariance properties isolate aspects of the data that should *not* be measured. They include *symmetry* (invariance to permutations of achievement vectors across people), *replication invariance* (invariance to replications of achievement vectors across people), *deprivation focus* (invariance to an increment in nondeprived achievements), and *poverty focus* (invariance to an increment in an achievement of a nonpoor person).

Next are the dominance properties that concern the aspects of the data that *should* be measured, and ensure that the poverty level responds appropriately to certain changes in the achievements. They include *weak monotonicity* (an increment in a single achievement cannot increase poverty), *weak transfer* (a progressive transfer among the poor arising from the same bistochastic matrix in each dimension cannot increase poverty), and *weak rearrangement* (a progressive transfer among the poor arising from an association decreasing rearrangement cannot increase poverty).[11]

Finally are the subgroup properties that connect poverty levels overall to levels obtained from data broken down by population subgroup or by dimension. Two of the key properties here are *subgroup decomposability* (overall poverty is a population weighted sum of the poverty levels in population subgroups) and *subgroup consistency* (if poverty rises in a population subgroup and stays constant in the remaining population, while subgroup population sizes are unchanged, then overall poverty must rise).

For the purposes of this paper, we will take the above set of four invariance properties, three dominance properties, and two subgroup properties as the nine *basic multidimensional properties*. Below we discuss another dominance property from Alkire and Foster (2011), namely, *dimensional monotonicity*, which requires poverty to fall as a result of an increment that removes at least one deprivation from among the poor. We now present the three additional properties of multidimensional measure – respectively, an invariance property, a subgroup property and a dominance property – that are the special concern of this paper.

3.1 **Ordinality**. The basic data used to construct the achievement matrix are typically derived from circumstances and conditions that are easy to describe and understand but have no natural metric in which to be measured. The numbers assigned to the various achievement levels (and deprivation cutoff) in this domain are in a real sense simply placeholders to convey information about how the levels stack up and, most importantly, whether they would be considered to be adequate or inadequate.[12] The underlying achievement levels, the

[11] Note that weak monotonicity and especially weak transfer are suited for measures using cardinal data, where the degree of change in a given dimension has meaning. For measures like M_0 using ordinal data, the size of increments or transfers cannot be meaningfully gauged and reflected through the measure's value. Instead, they satisfy the property with equality.

[12] In fact, categorical information is all that is necessary in the present context. Even if the deprived achievements cannot be ranked one against the other, and the same is true for the achievements in the non-

condition of deprivation, and its relative values in poverty, can be well understood and articulated even without the numbers of the representation being anything more than just placeholders. Note that this general line of argument may be true even for the cases where the variable has an "in-built" representation such as income or years of schooling, since the cardinalization that comes with the variable may not be the right one for reckoning gains and losses in the present context.[13]

We say that $(y'; z')$ is obtained from $(y; z)$ as an *equivalent representation* if there exist increasing functions f_j: $R_+ \rightarrow R_+$ for $j = 1,\ldots,d$ such that $y'_{ij} = f_j(y_{ij})$ and $z'_j = f_j(z_j)$ for all $i = 1,\ldots, n$. In other words, an equivalent representation assigns a different set of numbers to the same underlying basic data while preserving the original order. The methodology $\mathcal{M}_{k0}$ satisfies the following invariance property, which embodies the concern that the measure should be independent of the way the underlying data are represented.[14]

Ordinality (*O*): Suppose that $(y'; z')$ is obtained from $(y; z)$ as an equivalent representation. Then the methodology $\mathcal{M} = (\rho, M)$ satisfies $\rho(y'_i; z') = \rho(y_i; z)$, for all i, and $M(y'; z') = M(y; z)$.

To see that $\mathcal{M}_{k0}$ satisfies this property, note that the dimensions in which person i is deprived are unchanged between $(y'; z')$ and $(y; z)$, since the monotonic transformation ensures that $y'_{ij} < z'_j$ whenever $y_{ij} < z_j$. Consequently, the deprivation count is unchanged, which ensures that $\rho_k(y'_i; z') = \rho_k(y_i; z)$ for all i. It follows that the associated censored deprivation matrices are identical, so that their means are the same, and hence $M_0(y'; z') = \mathrm{M}_0(\mathrm{y}; z)$. Note that the since the gap and squared gap matrices are typically very different for equivalent representations, the methodology $\mathcal{M}_{k\alpha} = (\rho_k, M_\alpha)$ violates ordinality for $\alpha = 1$, $\alpha = 2$, and indeed any $\alpha > 0$. Each of these makes use of cardinal information on the depth of deprivations.

deprived category, one could use any numerical assignment that would correctly separate achievements into the deprived or non-deprived categories, with the deprivation cutoff being set at an appropriate value in between. The functions used below in the definition of equivalent representation need only preserve the categorical allocations.

[13] For a fuller treatment of scales and measurement see Stevens (1946), Sen (1973, 1997), Alkire et al (2014), and the references therein.

[14] We might imagine a weaker ordinality requirement that would only require the ordering, and not necessarily the measured level of poverty, to be preserved by equivalent representations.

3.2 **Dimensional Breakdown**. Multidimensional poverty by definition has multiple origins, and it is useful for policy purposes to have a method of gauging how each dimension contributes to overall poverty. For example, information on contributions of dimensional deprivations could help in the allocation of resources across sectors and the design of specific policies to address poverty; monitoring progress dimension by dimension can help clarify the underlying sources of progress.[15] A thoroughgoing decomposition of poverty by dimension would require every dimensional component to be a function of that dimension's distribution of achievements only, without reference to achievements in the other dimensions.[16] However, a person's deprivation in a dimension should only contribute to poverty when the person is *poor*, and this depends the person's achievements in *other* dimensions through the identification function. Consequently, we consider a less exacting form of breakdown by dimension that allows the component function in a given dimension to depend on the distribution of that dimension's achievements *and* on who is and is not poor, as determined by $\rho(y_i; z)$ for $i = 1,\ldots,n$. Stated differently, when we limit consideration to the set Y_r of achievement matrices having exactly the same poverty status vector r, overall poverty can be expressed as a weighted sum of dimensionally determined components.[17] As before, for any given dimension j, let Y_{rj} denote the set of all achievement distribution vectors $y_{.j}$ from some y in Y_r.

Dimensional breakdown (*DB*): For any given poverty status vector r, there exist $v_j > 0$ summing to one, and functions $m_{.j}$: $Y_{rj} \times R^d_{++} \rightarrow R$ for $j = 1,\ldots, d$, such that

$$M(y; z) = v_1\, m_{.1}(y_{.1}; z) + \ldots + v_d\, m_{.d}(y_{.d}; z) \qquad (3)$$

for all y in Y_r.

In words, after identification has taken place and the poverty status of each person has been fixed, multidimensional poverty can be expressed as a weighted sum of dimensional components. The contribution of the deprivations in the jth dimension to overall poverty can then be viewed as $v_j\, m_{.j}(y_{.j}; z)/M(y; z)$.

[15] Each of these examples is used in practice: for example in Colombia and in the Brazilian state of Minas Gerais. Naturally the translation from measure to policy response requires additional analysis, as deprivations are often interconnected.

[16] An unlimited decomposability property of this type has been studied by Chakravarty, Mukherjee, and Ranade (1998), who call it "factor decomposability." See also Chakravarty and Silber 2008, Chakravarty 2010.

[17] This property allows the functional form of the breakdown to vary for every set of distributions having a different set of the poor – a less stringent and more general assumption than a full dimensional decomposition that requires the same functional form across all the subsets.

For example, in the case of the adjusted headcount ratio $\mathcal{M}_{k0} = (\rho_k, M_0)$, expression (2) yields its version of equation (3), namely

$$M_0 = [(w_1/d)\,|\,g^0_{\cdot 1}(k)\,|\,(nw_1)+\ldots+(w_d/d)\,|\,g^0_{\cdot d}(k)\,|\,(nw_d)]$$

$$= (w_1/d)\,H_1 +\ldots+ (w_d/d)\,H_d$$

where the dimensional weight is $v_j = w_j/d,$ or the value of deprivation j over the sum of the deprivation values, and the dimensional component is $m_{\cdot j}(y_{\cdot j}; z) = H_j = g^0_{\cdot j}(k)/(nw_j)$, where H_j is a censored headcount ratio or the percentage of the population that is both deprived in dimension j and poor. Notice that H_j depends on the distribution of the other dimensional achievements, since all are needed to determine whether a person is poor. However, the entries in column $g^0_{\cdot j}(k)$ can be expressed as $g^0_{ij}(k) = g^0_{ij}\rho_k(y_i; z)$, and hence depend on the other dimensional achievement levels *only through the identification function*. This ensures that when we restrict consideration to distributions having the same fixed poverty status vector r, the term reduces to $g^0_{ij}(k) = g^0_{ij} r_i$ and hence depends only on the achievements in dimension j, as required by dimensional breakdown.[18]

3.3 **Dimensional Transfer**. Transfer properties are motivated by the idea that poverty should be sensitive to the level of inequality among the poor, with greater inequality being associated with a higher (or at least no lower) level of poverty.[19] But which notion of inequality should be used in the multidimensional context? As noted in the introduction, there are two concepts in common use, one due to Kolm (1977) and another due to Atkinson and Bourguignon (1982). The first is based on a definition of a progressive transfer as a "common smoothing", whereby each dimensional distribution is transformed using the same bistochastic matrix. However, for this form of inequality to be meaningful, each dimensional variable would need to exhibit properties that are at odds with the ordinality axiom.[20]

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[18] It can likewise be shown that every adjusted FGT methodology $\mathcal{M}_{k\alpha} = (\rho_k, M_\alpha)$ for $\alpha > 0$, as presented in Alkire and Foster (2011), satisfies dimensional breakdown.

[19] See Sen (1976), Foster and Sen (1997), and Alkire et al (2014).

[20] The transformed achievement levels in a dimension are weighted averages of initial levels, and hence depend on the cardinal representation of variables, which goes against the ordinality axiom. In particular, after such a transformation, a person might be seen as poor under one cardinalization and nonpoor under a second. Measures applicable to ordinal variables cannot depend on the inequality level arising from a given cardinalization and, like M_0, are independent of this form of inequality among the poor.

The second inequality concept is based on a specialized transfer called a rearrangement, in which two persons switch achievements in certain dimensions. The role of progressive transfer in this context is played by an *association decreasing rearrangement*, in which the achievement vectors of the two persons are initially ranked by vector dominance (so that one person has no less in each dimension than the other person and more in one) and then after the rearrangement their achievement vectors cannot be ranked (so that one person has more in one dimension and the other has more in a second dimension). This transformation can be interpreted as a progressive transfer in that it transforms an initial "spread" between two persons – a spread represented by the dominance between achievement vectors – into a moderated situation where neither person has unambiguously more than the other. The overall achievement levels in society are unchanged, but the correlation between them (and hence inequality) has been reduced.

Since this form of transfer involves a permutation, and not an algebraic averaging, of two persons' dimensional achievements it can be applied to ordinal data and is in principle consistent with property *O* for poverty measures. The associated axiom for multidimensional poverty measures typically requires the two persons involved in the rearrangement to be poor. For example, the *weak rearrangement* axiom, as defined in Alkire and Foster (2011), requires poverty not to rise as a result of an association decreasing rearrangement among the poor. Note that this axiom – like all related axioms in the literature – is weak in that it does *not* require poverty to strictly fall. It rejects the most problematic measures for which poverty can be "alleviated" by increasing inequality among the poor, but at the same time allows measures to be entirely insensitive to progressive rearrangements among the poor.[21]

A natural question to ask is whether an alternative version of this transfer axiom can be formulated that would, in certain circumstances, require poverty to strictly fall in response to a decline in inequality among the poor. One minimalist approach is to restrict consideration to cases where the association decreasing rearrangement among the poor involves achievement levels that are on either side of deprivation cutoffs - thus affecting the distribution of deprivations as well. A *dimensional rearrangement among the poor* is an association decreasing rearrangement among the poor (in achievements) that is simultaneously an association decreasing rearrangement in *deprivations*. In other words, the

[21] Such is the case of the headcount ratio H and the adjusted headcount ratio M_0.

initial deprivation vectors (and achievement vectors) are ranked by vector dominance, while the final deprivation vectors (and achievement vectors) are not.[22] The extra condition ensures that the person with lower level of achievements is actually deprived in some dimensions for which the other person is not, and that through the rearrangement one or more of these deprivations (but not all) are traded for non-deprived levels. The following transfer property for multidimensional poverty measures requires poverty to decrease when there is a dimensional rearrangement among the poor.

Dimensional transfer (*DT*): If y' is obtained from y by a dimensional rearrangement among the poor, then $M(y';z) < M(y;z)$.

This axiom does not apply to cases where the association decreasing rearrangement leaves deprivations unaffected; instead it requires the two persons to switch deprivations *as well as* achievements. Note that this axiom is analogous to the axiom of dimensional monotonicity found in Alkire and Foster (2011), which provides conditions under which a decrement in a dimensional achievement of a poor person must strictly raise the poverty level, namely, whenever the deprivation cutoff is crossed and the person becomes deprived in that dimension. Poverty must strictly rise as a result of such a *dimensional decrement among the poor* - which alters the achievement vector of a poor person so that there is vector dominance (upwards) in deprivations as well as vector dominance (downwards) in achievements.

A dimensional rearrangement among the poor does not affect the number of poor persons, and neither does a dimensional decrement among the poor. Consequently the headcount ratio $\mathcal{H}_k = (\rho_k, H)$ violates both dimensional transfer and dimensional monotonicity. In contrast, a dimensional increment among the poor decreases the average intensity of poverty A, and hence $M_0 = HA$ which ensures that the adjusted headcount ratio $\mathcal{M}_{k0} = (\rho_k, M_0)$ satisfies dimensional monotonicity. But since a dimensional rearrangement among the poor leaves H and A unchanged, $\mathcal{M}_{k0}$ just fails to satisfy the dimensional transfer axiom. The adjusted headcount ratio is insensitive to this form of inequality among the poor. The next section explores the possibility of constructing alternative measures that satisfy the dimensional transfer property.

[22] Note that the vector dominance in deprivations must be in the converse direction to the vector dominance in achievements. The person with lower achievements has additional deprivations.

4. **New Measures from Old**

The task at hand is to construct new methodologies that follow $\mathcal{M}_{k0}$ in being able to be applied to ordinal data, but unlike $\mathcal{M}_{k0}$ are sensitive to inequality among the poor. We maintain the dual cutoff approach to identification ρ_k, which is reasonably flexible and consistent with property O, and search for a multidimensional measure M whose methodology $\mathcal{M} = (\rho_k, M)$ satisfies properties O and DT. We begin by altering the adjusted headcount measure to obtain such a measure, and then expand the range of possibilities using a novel way of constructing multidimensional poverty measures from unidimensional poverty measures – a process that is of interest in its own right.

Alkire and Foster (2011) applied a simple power transformation to the individual poverty function from their cardinal measures, M_α for $\alpha > 0$, to obtain altered measures that would be sensitive to inequality across dimensions.[23] When applied to M_0, the same transformation yields a methodology that satisfies both O and DT. For any power $\gamma > 0$, let $s^\gamma(k)$ be the vector whose i^{th} entry is $\rho_k(y_i;z)(s_i)^\gamma$, so that $s_i^\gamma(k)$ is $(s_i)^\gamma$ when person i is poor and 0 when person i is not. For example, $\gamma = 2$ would produce $s^2(k)$, the censored vector of squared deprivation scores, while $\gamma = 1$ would yield the original censored vector of deprivation scores $s^1(k) = s(k)$. Now define $M_0^\gamma = \mu(s^\gamma(k))$, and note that for $\gamma = 1$, the measure reduces to the usual adjusted headcount ratio, while for $\gamma > 1$ it places disproportionate emphasis on persons with the highest deprivation scores.[24] We obtain the following result.

Proposition 1 For every $\gamma > 1$, methodology $\mathcal{M}_0^\gamma = (\rho_k, M_0^\gamma)$ satisfies properties O and DT.

Proof Suppose that $(y'; z')$ is obtained from $(y; z)$ as an equivalent representation. It is clear that $(y'; z')$ and $(y; z)$ have the same deprivation matrix, and hence the same deprivation counts for all i, from which it follows that $\rho_k(y'_i; z') = \rho_k(y_i; z)$. Hence the censored deprivation scores are the same, which implies that $M_0^\gamma(y'; z') = M_0^\gamma(y; z)$ for every $\gamma > 1$. Thus, $\mathcal{M}_0^\gamma$ satisfies property O.

[23] See Alkire and Foster (2011a) p. 485.

[24] In the case of union identification and equal valued deprivations, M_0^γ corresponds to the measure of social exclusion proposed in Chakravarty and D'Ambrosio (2006) and used in Jayaraj and Subramanian (2010). As we note below, it also has obvious links to the FGT class of unidimensional measures.

Now suppose that y' is obtained from y by a dimensional rearrangement among the poor, where h is the better off poor person and i is the worse off, and j is the dimension in which h gives up a deprivation to i. We want to show that $M_0^\gamma(y'; z) < M_0^\gamma(y; z)$, or equivalently $\mu(s^\gamma(k)') < \mu(s^\gamma(k))$. By definition, only columns h and i of the censored deprivation matrices are altered by the rearrangement, hence $s(k)'$ and $s(k)$ only differ in coordinates h and i. As $M_0(y'; z) = M_0(y; z)$ it follows that $\mu(s(k)') = \mu(s(k))$ and so $s_h(k)' + s_i(k)' = s_h(k) + s_i(k)$, which implies that there is a constant Δ such that $s_h(k)' = s_h(k) + \Delta$ and $s_i(k)' = s_i(k) - \Delta$. The rearrangement has person i shifting a deprivation in dimension j to person h, and hence $\Delta = w_j /d > 0$. Moreover, by the vector dominance of i's deprivation vector over h's deprivation vector, we know that $s_h(k) < s_i(k)$; and by the subsequent lack of vector dominance it follows that $s_h(k)'$ and $s_i(k)'$ are both strictly between $s_h(k)$ and $s_i(k)$. It therefore follows that $s(k)'$ is obtained from $s(k)$ by a progressive transfer (in this case of the deprivation score) from person i to person h and, since $\gamma > 1$, it must be true that $\mu(s^\gamma(k)') < \mu(s^\gamma(k))$ as we set out to show. Thus, $\mathcal{M}_0^\gamma$ satisfies property *DT*. ■

The above proof shows how a dimensional rearrangement among the poor across two achievement matrices becomes a progressive transfer among the poor for the associated deprivation score vectors, which has the effect of lowering poverty when it is measured as the average of the deprivation scores to the power $\gamma > 1$. Note that a person's deprivation score resembles the normalized poverty gap in the unidimensional world, so that M_0^γ has a form analogous to an *FGT* index, where γ is the power on the normalized gaps. We can build upon this insight to show how other multidimensional poverty measures satisfying *O* and *DT* might be derived from unidimensional poverty measures. But before doing this, we pause to review the basic structure of unidimensional methods, along with some recent elaborations.

A digression on unidimensional poverty measurement

A unidimensional poverty measure is a real-valued mapping $P: X \times R_+ \rightarrow R$ from the set X of nonnegative income distributions of all population sizes and the set R_+ of potential poverty lines. For any given income distribution $x = (x_1,x_2,\ldots,x_n)$ in X and poverty standard $\pi \geq 0$, the value $P(x; \pi)$ is interpreted as the level of poverty in x given π. Common examples include the *FGT* class, the Watts measure, and the Sen measures; properties for unidimensional poverty measures include invariance properties such as *symmetry*, *replication invariance*, *scale invariance*, and *focus*, dominance properties like *monotonicity* and *transfer*, and the subgroup properties of *subgroup consistency* and *subgroup decomposability*.[25] For simplicity, we will focus on subgroup decomposable, and hence subgroup consistent, measures.

In addition to specifying an aggregate measure, a poverty methodology must determine the precise criterion for identifying the poor. In the unidimensional environment, this usually begins with the selection of the poverty cutoff π designed to separate the persons targeted by the methodology and the remaining population. But as stressed by Donaldson and Weymark (1986), selecting the cutoff is not quite enough. The method must indicate whether the poor have incomes *below* the cutoff (the traditional "weak" definition that treats the cutoff as a minimum level) or incomes *below or equal to* the cutoff (the less traditional "strong" definition found in the literature following Sen (1976)).[26] To convey which identification method is being used, we can specify an *identification function* φ: $R_+ \times R_+ \rightarrow \{0,1\}$, whose value $\varphi(x_i; \pi)$ indicates when person i with income x_i is poor given poverty cutoff π. In particular, a value of $\varphi(x_i; \pi) = 1$ indicates that i is poor, while a value of $\varphi(x_i; \pi) = 0$ indicates that i is nonpoor. The function φ has two possible forms: (i) the *weak* identification function, denoted by φ_w, and defined by $\varphi_w(x_i; \pi) = 1$ for $x_i < \pi$, and $\varphi_w(x_i; \pi_I) = 0$ for $x_i \geq \pi$; or (ii) the *strong* identification function, denoted by φ_s, and defined by $\varphi_s(x_i; \pi) = 1$ for $x_i \leq \pi$, and $\varphi_s(x_i; \pi_I) = 0$ for $x_i > \pi$. The strong identification function expands the set of the poor to include persons at the poverty line.

With the help of this notation, the standard subgroup decomposable measures can be written as

$$P(x; \pi) = (1/n)\ \Sigma_i\, p(x_i; \pi)\varphi(x_i; \pi)$$

where p: $R_+ \times R_+ \rightarrow R$ is a *poverty value function* that gauges a person's poverty level when poor (but is ignored when the person is not poor) and φ is an identification function.[27] For example, the headcount ratio can be written using the constant poverty value function $p(x_i; \pi)$

[25] For definitions, see for example Foster and Sen (1997) or Foster et al (2013).
[26] See Donaldson and Weymark (1986). This distinction is relevant when the poverty measure has a discontinuity at the poverty line, such as exhibited by the headcount ratio.
[27] This is analogous to the decomposable form given in Atkinson (1987) and Shorrocks and Foster (1991), but with separate terms for aggregation and identification. This representation implies that *P* is the normalized version of the measure, so that *P* = 0 whenever no one is poor. Any subgroup decomposable measure can be so normalized by subtracting out the nonpoor "poverty level".

= 1, while the remaining *FGT* measures can use $p(x_i; \pi) = ((\pi - x_i)/\pi)^\alpha$ for $\alpha > 0$. Notice that the choice of φ_w or φ_s has an impact on the measured level of poverty whenever $p(\pi; \pi) > 0$ (as with the headcount ratio), while the choice has no impact whenever the measure has $p(\pi; \pi) = 0$ (as with the remaining *FGT* indices).

The above representation emphasizes that there are two distinct roles for π in unidimensional poverty measurement: first as the poverty cutoff that is used in the identification of the poor via φ; second as the poverty standard used in the valuation of their poverty via p. The traditional approach does not differentiate between the two and instead uses the same value in both roles. Recent work has highlighted instances where it may be very useful to distinguish between the two, with π_I denoting the poverty cutoff from identification, π_A denoting the poverty standard from aggregation, and $\pi_I \leq \pi_A$ reflecting the practical requirement that anyone identified as being poor should not have an income level beyond the poverty standard used in aggregation.[28] For example, when evaluating the extent of ultra-poverty a lower poverty cutoff π_I must be used to identify this group of most deprived poor. However, it does not follow that the poverty standard π_A used in gauging the intensity of aggregate poverty must also be lowered - a change that would make their poverty seem less intense. With $\pi_I < \pi_A$, an ultra-poor poverty cutoff could be used without compromising the prevailing poverty standard. Following Foster and Smith (2014) we differentiate between π_I and π_A to obtain greater flexibility in targeting and measurement. In symbols, a *unidimensional poverty methodology* is denoted by $\mathcal{P} = (\varphi, P)$, where $\varphi(x_i; \pi_I)$ is the *identification function* using the *poverty cutoff* π_I for identification and

$$P(x; \pi_A) = (1/n)\, \Sigma_i\, p(x_i; \pi_A)\varphi(x_i; \pi_I) \qquad (4)$$

is the *poverty measure* using the *poverty standard* $\pi_A \geq \pi_I$ for aggregation.[29]

The standard properties for unidimensional poverty measures apply directly to the broader context considered here (where the poverty standard may differ from the poverty cutoff), so long as it is remembered that the relevant set of the poor (or nonpoor) is determined by φ evaluated at π_I and not π_A. These properties include the invariance properties of *symmetry*,

[28] See Foster and Smith (2013).
[29] The approach is generalized to non-decomposable measures in Foster and Smith (2013). However, for simplicity of presentation, and because of the general importance of subgroup decomposability in practical application, we restrict consideration to decomposable measures here.

replication invariance, and *focus*, and the subgroup properties of *subgroup consistency* and *subgroup decomposability*. For the purposes of this paper we will call these three invariance properties and two subgroup properties the *basic unidimensional properties*, and focus on methodologies satisfying them.[30]

Two additional dominance properties will now be defined, which make use of the following forms of basic distributional changes. We say that x' is obtained from x by a *decrement among the poor* if there is a person i with $\varphi(x_i; \pi_I) = 1$ such that $x_i' < x_i$, while for all $h \neq i$ we have $x_h' = x_h$. In words, such a decrement involves a poor person losing income while all other incomes are unchanged. We say that x' is obtained from x by a *progressive transfer among the poor* if there are two persons h, i with $\varphi(x_h; \pi_I) = \varphi(x_i; \pi_I) = 1$ such that $x_h' - x_h = x_i - x_i' = \Delta > 0$ where $x_i - x_j > \Delta$, while for all other i' we have $x_{i'}' < x_{i'}$. In words, such a transfer involves a richer poor person giving income to an even poorer person, but not so much that they switch incomes.

Monotonicity: If x' is obtained from x by an decrement among the poor, then
$P(x'; \pi_A) > P(x; \pi_A)$
Transfer: If x' is obtained from x by a progressive transfer among the poor, then
$P(x'; \pi_A) < P(x; \pi_A)$

The properties specify that the poverty methodology must register an increase in poverty when there is a decrement among the poor, and a decrease in poverty when there is a progressive transfer among the poor.

From unidimensional to multidimensional

Now that we have reviewed the structure of unidimensional poverty measurement, we return once again to the multidimensional environment where a useful way of obtaining multidimensional poverty methodologies from unidimensional methodologies can now be described. The process begins by defining a matrix that is complementary to the deprivation matrix g^0 as it indicates when persons are *not* deprived. The *attainment matrix*, denoted by a^0,

[30] Of course, all unidimensional methodologies considered here have the form (4) and hence satisfy subgroup decomposability and subgroup consistency by construction.

is the matrix having the typical element $a_{ij}^0 = w_j - g_{ij}^0$ for $i = 1,\ldots, n$ and $j = 1,\ldots, d$. In other words, $a_{ij}^0 = w_j$ whenever i is *not* deprived in j, and $a_{ij}^0 = 0$ whenever i *is* deprived in j. The *attainment count vector*, denoted by a, is the vector defined by $a_i = a_{i1}^0 +\ldots+ a_{id}^0$ for each $i = 1,\ldots,n$. In other words, it gives an aggregate value of attainment for each person. Attainment counts can range between 0 and d, and together with deprivation counts they sum to d (so that $a_i + c_i = d$ for all i).

Now let $\mathcal{P} = (\varphi, P)$ be a unidimensional poverty methodology with poverty cutoff π_I and poverty standard π_A satisfying $0 \leq \pi_I \leq \pi_A \leq d$. Define the associated multidimensional poverty methodology $\mathcal{M}_{\mathcal{P}} = (\rho_{\varphi}, M_P)$ by $\rho_{\varphi}(y_i; z) = \varphi(a_i; \pi_I)$ and $M_P(y; z) = P(a; \pi_A)$ where a is the attainment count vector associated with y given z. In other words, $\mathcal{M}_{\mathcal{P}}$ applies the unidimensional methodology $\mathcal{P}$ to the attainment count distribution. In particular, $\mathcal{M}_{\mathcal{P}}$ identifies person i as being poor in y if $\varphi(a_i; \pi_I) = 1$; and it measures poverty in y as the unidimensional poverty level $P(a; \pi_A)$ in a, given the poverty standard π_A. The resulting multidimensional poverty methodology $\mathcal{M}_{\mathcal{P}}$ will be called an *attainment count methodology* while the associated process of applying $\mathcal{P}$ to obtain $\mathcal{M}_{\mathcal{P}}$ will be called the *attainment count transformation*.

It is easy to see that the attainment count methodology uses a dual cutoff identification from Alkire and Foster (2011), where the poverty cutoff is given by $k = d - \pi_I$. For example if $\varphi = \varphi_s$ is being used in $\mathcal{P}$, then since $\rho_{\varphi}(y_i; z) = \varphi_s(a_i; \pi_I)$, it follows that person i is poor whenever $a_i \leq \pi_I$, hence $d - c_i \leq d - k$ or $c_i \geq k$ as required by the standard dual cutoff identification function ρ_k. On the other hand, if $\mathcal{P}$ uses $\varphi = \varphi_w$, then the identification function ρ_{φ} yields an alternative dual cutoff form ρ_k' where once again $k = d - \pi_I$, but a person is considered poor if $c_i > k$.[31] To see how the aggregation in $\mathcal{M}_{\mathcal{P}}$ is derived from $\mathcal{P}$, we provide several basic examples.

Example 1: *Headcount Ratio* Suppose that $\varphi = \varphi_s$ and $P = P_0$, where P_0 is the unidimensional headcount ratio. Fix $\pi_A = d$ and pick any π_I satisfying $0 \leq \pi_I < d$. Then $\mathcal{M}_{\mathcal{P}} = (\rho_k, H)$ where ρ_k is the dual cutoff identification function given poverty cutoff $k = d - \pi_I$, and $H(y; z) = (1/n)\ \Sigma_i$

[31] Both versions of the dual cutoff approach are considered by Alkire and Foster (2011) although they focus on the version based on the inequality $c_i \geq k$.

$\rho_k(y; z)$ is the usual multidimensional headcount ratio. The case $\pi_I = 0$ (and hence $k = d$) corresponds an intersection identification, since being poor means being deprived in all dimensions at once and having no attainments at all. Likewise, π_I close to d (and hence k close to 0) yields the union identification, in which the poor have at least a single deprivation and an attainment count below d.[32] Notice that since P is the headcount ratio, the poverty standard π_A is superfluous here.

Example 2: Adjusted Headcount Ratio Suppose that $\varphi = \varphi_s$ and $P = P_1$, where P_1 is the unidimensional poverty gap measure. Fix $\pi_A = d$ and pick any π_I satisfying $0 \leq \pi_I < d$. Then $\mathcal{M}_{\mathcal{P}} = (\rho_k, M_0)$ where the identification function is ρ_k with $k = d - \pi_I$ while M_0 is the adjusted headcount ratio of Alkire and Foster (2011). To see this, note that when the poor are identified using $\varphi_s(a_i; \pi_I)$ and measured using $P_1(a; \pi_A)$, then the overall poverty level is found by averaging the terms $\varphi_s(a_i; \pi_I)(\pi_A - a_i)/\pi_A$ across all $i = 1,\ldots,$n. But for every poor person i (with $a_i \leq \pi_I$, and hence $c_i \geq k$) this term reduces to $c_i/d = c_i(k)/d$, while for all nonpoor i (with $a_i > \pi_I$, and hence $c_i < k$) it is $0 = c_i(k)/d$. The average of $c_i(k)/d$ across all i is the same as the average across all entries in the censored deprivation matrix $g^0(k)$ and hence the poverty measure is just the adjusted headcount ratio $M_0 = \mu(g^0(k))$.[33]

Example 3: *An Alternative Headcount Ratio* Suppose that $\varphi = \varphi_w$ and $P = P_0$ is the unidimensional headcount ratio. Pick any π = satisfying $0 < \pi \leq d$ and set $\pi_I = \pi_A = \pi$. Then $\mathcal{M}_{\mathcal{P}} = (\rho_k', H)$ where H is the usual multidimensional headcount ratio and ρ_k' is the alternative dual cutoff identification function given poverty cutoff $k = d - \pi$. Here the case $\pi = d$ (and hence $k = 0$) corresponds to union identification, since i is poor whenever $c_i > 0$ signifying at least one deprivation. Likewise, π close enough to 0 (and hence k close enough to d) yields the intersection identification, in which the poor must be deprived in all dimensions and have no attainments at all.[34] Nothing would change if π_A were strictly larger than if $\pi_A = \pi$, since P is the headcount ratio and does not utilize the poverty standard. Real world examples of $(\rho_k'$,

[32] More precisely, union identification is obtained when $\pi_I = d - w_{min}$ or above, so that $k = w_{min}$ or below, where w_{min} is the lowest deprivation value.
[33] This is the form used in the UNDP's Multidimensional Poverty Index (MPI) and in Colombia's official multidimensional poverty measure. See also Alkire and Santos (2010).
[34] More precisely, intersection identification is obtained when $\pi = w_{min}$ or below, so that $k = d - w_{min}$ or above, where w_{min} is once again the lowest deprivation value.

H) include the multidimensional headcount ratio reported as part of Mexico's official poverty methods.[35]

Example 4: An Alternative Adjusted Headcount Ratio Suppose that $\varphi = \varphi_w$ and pick $\pi_I = \pi_A = \pi$ where $0 < \pi \leq d$, so that $P = P_1$ is the traditional unidimensional poverty gap. Then $\mathcal{M}_p = (\rho_k', \mathrm{M}_0')$, where the identification function is ρ_k' with $k = d - \pi$ and the measure is $M_0'(y;z) = (1/n)\Sigma_i\, (c_i'/d')\, \rho_k'(y_i; z)$, where $c_i' = c_i - k$ is the deprivation count above the poverty cutoff for person i, and d' is the maximum possible deprivation count above the poverty cutoff. In words, $M_0'(y;\, z)$ is an alternative adjusted headcount ratio that measures a poor person's intensity of deprivation using the net deprivation share c_i'/d'. To see this, note that $P_1(a;\, \pi) = (1/n)\ \Sigma_i\ \varphi_w(a_i;\, \pi)(\pi - a_i)/\pi$ where $(\pi - a_i)/\pi = (d\text{-}k\text{-}a_i)/(d\text{-}k) = (c_i\text{-}k)/(d\text{-}k) = c_i'/d'$ and $\varphi_w(a_i;\, \pi) = \rho_k'(y_i;\, z)$, so that clearly $P_1(a;\, \pi) = M_0'(y;\, z)$. In the Mexican example, the deprivation values $w_1 = 7/2$ and $w_j = 7/12$ for $j = 2,\ldots,7$ and the poverty cutoff $k = 7/2$ along with the identification condition $c_i > k$ ensures that all poor persons are deprived in the first dimension, which is income, and at least one additional non-income dimension. The intensity of a poor person's deprivation is measured by the share of all possible non-income deprivations a person has, while a nonpoor person has zero intensity. Averaging across all persons yields the alternative version of the adjusted headcount ratio M_0'.

The above examples show that several key multidimensional methodologies used in practice are in fact attainment count methodologies, including the headcount and adjusted headcount methodology from Alkire and Foster (2011), employed in the *MPI* and Colombia's official measure, and the alternative headcount and adjusted headcount methodologies that underlie Mexico's official measure. Note that while all of these examples satisfy ordinality, none satisfies the dimensional transfer property. Our goal in reviewing unidimensional measurement and constructing attainment count methodologies was to obtain measures satisfying conditions *O* and *DT*. The following result demonstrates just how easy this is.

[35] The Mexican technology has seven dimensions with effective deprivation values of $w_1 = 7/2$ and $w_j = 7/12$ for $j = 2,\ldots,7$ and a poverty cutoff of $k = 7/2$. See CONEVAL (2009).

Proposition 2 Let $\mathcal{P}$ be a unidimensional poverty methodology (4) that satisfies the five basic unidimensional properties. Then the attainment count methodology $\mathcal{M}_{\mathcal{P}}$ satisfies the nine basic multidimensional properties. Moreover:

(i) $\mathcal{M}_{\mathcal{P}}$ satisfies dimensional monotonicity whenever $\mathcal{P}$ satisfies monotonicity

(ii) $\mathcal{M}_{\mathcal{P}}$ satisfies the dimensional transfer property whenever $\mathcal{P}$ satisfies the transfer property.

Proof: Let y and y' denote two distributions and let a and a' denote their respective attainment vectors, so that where P is the poverty measure from $\mathcal{P}$ and M_P is the measure from the attainment count methodology $\mathcal{M}_{\mathcal{P}}$ defined using $\mathcal{P}$, we have $M_P(y; z) = P(a; \pi_A)$ and $M_P(y'; z) = P(a'; \pi_A)$. Given that $\mathcal{P}$ satisfies the basic properties for unidimensional poverty methodologies, namely, symmetry, replication invariance, and the focus axiom, it is an easy matter to show that $\mathcal{M}_{\mathcal{P}}$ likewise satisfies the basic properties for multidimensional poverty methodologies, namely symmetry, replication invariance, and the poverty focus axiom. For if y' is obtained from y by a permutation, then given the definition of the attainment count vector, a' must be obtained from a by a permutation, so that by symmetry of $\mathcal{P}$ it follows that $P(a'; \pi_A) = P(a; \pi_A)$. By definition, then, $M_P(y'; z) = M_P(y; z)$, which shows that $\mathcal{M}_{\mathcal{P}}$ satisfies symmetry. An entirely analogous argument establishes that $\mathcal{M}_{\mathcal{P}}$ satisfies replication invariance. For poverty focus, suppose that y' is obtained from y by a simple increment among the nonpoor. This means that there is a single entry y_{ij} such that: the remaining entries in y' and y are the same; $y'_{ij} > y_{ij}$; and person i is nonpoor in y. The latter condition translates to $\rho_{\varphi}(y_i; z) = 0$ and hence $\varphi(a_i; \pi_I) = 0$, while the previous two ensure that $a'_i \geq a_i$. By the focus axiom for $\mathcal{P}$, it follows that $P(a'; \pi_A) = P(a; \pi_A)$ and so by definition $M_P(y'; z) = M_P(y; z)$. Hence $\mathcal{M}_{\mathcal{P}}$ satisfies the poverty focus axiom.

Now to show (i), if y' is obtained from y by a dimensional decrement among the poor, there is a single entry a_i such that: the remaining entries in a' and a are the same; $a'_i < a_i$; and i is poor in y according to $\mathcal{M}_{\mathcal{P}}$, and hence in a according to $\mathcal{P}$. (To be finished)

5. Possibilities

The previous section provided two ways of constructing intuitive multidimensional poverty indices satisfying dimensional transfer – a first approach used in Alkire and Foster (2011a) and other counting papers, which raises to a power an individual's poverty function before averaging across people; and a second approach, that converts everything into attainments – that are zero if the achievement level is below the deprivation cutoff and one if it is equal or above – constructs a distribution of attainment counts and applies a unidimensional poverty measure that satisfies the transfer property. However, we also note that although each of the examples we constructed to satisfy the dimensional transfer property, not one satisfies dimensional breakdown. Our next theorem offers insight as to why this is true.

Proposition 3: Then there is no multidimensional poverty methodology $\mathcal{M} = (\rho, M)$ satisfying symmetry, dimensional breakdown and dimensional transfer.

Proof: The assumption ensures that a switch among the poor can be constructed. Consequently, let distribution y'' be obtained from y' by a nontrivial progressive transfer or switch among the poor. By symmetry, without loss of generality we can let person 1 be the giver of the transfer and person 2 be the receiver. Let j' denote the dimension in which the switch takes place. In y'', replace y''_{1j} with y''_{2j} for all $j \neq j'$ to obtain achievement matrix x''. Similarly in y', replace y'_{1j} with y'_{2j} for all $j \neq j'$ to obtain achievement matrix x'. Clearly both y'' and y' have the same poverty status vector r, and since $y'_1 \geq y'_2$ while ρ is monotonic, so do x'' and x'. By dimensional breakdown, there exist weights $v_j > 0$ summing to 1 and functions $m_{.j}$: $Y_{rj} \times R^d_{++} \rightarrow R$ for j = 1,…, d, such that (3) holds for all y in Y_r, including y', y'', x' and x''. Applying (3) to y'' and y' yields $M(y';z) - M(y'';z) = v_{j'} (m_{j'} (y'_{.j'}; z_{j'}) - m_{j'} (y''_{.j'}; z_{j'}))$ while applying (3) to x'' and x' yields $M(x';z) - M(x'';z) = v_{j'} (m_{j'} (x'_{.j'}; z_{j'}) - m_{j'} (x''_{.j'}; z_{j'}))$. But notice that by construction $y'_{.j'} = x'_{.j'}$ and $y''_{.j'} = x''_{.j'}$ and so $M(y';z) - M(y'';z) = M(x';z) - M(x'';z)$. Finally, notice that x' is simply a permutation of x'' (between persons 1 and 2) and so $M(x';z) - M(x'';z) = 0$. This implies that $M(y';z) = M(y'';z)$ and hence dimensional transfer is violated.

Given that it is necessary to choose between measures that satisfy dimensional transfer, and those which can be broken down by dimension, and given that both properties are arguably important, how are empirical and operational studies to proceed? The first option is to employ the class of measures that respect dimensional breakdown, and to supplement these with associated inequality measures including poverty measures that satisfy dimensional transfer. The second is to supplement measures that incorporate inequality among the poor, with dimensional analyses that draw on the original (censored) deprivation or attainment matrix.

Whilst both should be explored, we favour the first route in applied work for several reasons. Dimensional breakdown enriches the informational content of poverty measures, enabling them to be used to tailor policies to the composition of poverty, to monitor changes by dimension, and to compare across time and space. Poverty reduction in measures respecting dimensional breakdown can be accounted for in terms of changes in deprivations among the poor, and analysed by region and dimension. This creates positive feedback loops that reward effective policies. The distribution sensitive measures also enrich the information in important ways: they illuminate the patterns of between-group and within-group inequality among the poor. But it may be possible to shed light on inequality in other ways. This section sketches out some options for doing so. Which methodology is followed depends upon the purpose of the exercise and the pertinent questions for analysis.

A basic but by no means un-illuminating option is to describe subsets of poor people having mutually exclusive and collectively exhaustive graded bands of deprivation scores. Thus when the poverty cutoff is one-third, as it is for the global MPI (Alkire and Santos 2010), it may be revealing to show which percentage of the poor have deprivation scores whose values fall in the band of 33-39% of deprivations, 40-49%, and so on to 100%. The percentage of people who experience different gradients of poverty across regions and time can be compared to see whether inequality is diminishing or advancing.[36] The comparisons can be enriched by applying dimensional breakdown to the subgroup experiencing each intensity band, and examining the dimensional composition of poverty experienced by those having different ranges of deprivation scores. And stochastic dominance can be explored.

A second option is to report two poverty measures for a given definition of poverty and dataset, one respecting DB and the other, DT. For example, one could report M_0 and M_0' - that is, the AF adjusted headcount ratio and the associated Squared Breadth measure. Differences in ordinal rankings between the measures expose different structures of inequality among the poor. Recall that the inequality measure associated with the FGT class of measures when alpha = 2 is the squared coefficient of variation, which is decomposable by population subgroups. The squared coefficient of variation is the average squared difference

[36] For empirical examples see Alkire Roche Seth and Sumner 2013, who compare countries across four gradients of poverty, and Alkire and Seth 2013, who show transitions across population subgroups of subgroup members experiencing different intensity bands of poverty.

between each poor people's income and the mean incomes of the poor, as a share of squared mean incomes among the poor. In a multidimensional setting, the squared coefficient of variation represents inequality across the deprivation (or attainment) scores of the poor. When the headcount ratio and the income poverty gap are held constant, then the FGT-2 measure and hence the M_0' measure varies with the squared coefficient of variation among the poor.

But if the companion measure to M_0 is used mainly to supplement M_0 with information on inequality among the poor, why should M_0' be used rather than another distribution sensitive measure – or indeed a direct measure of inequality? Why is the squared coefficient of variation – a relative measure – more appropriate than the two Theil indices, the Atkinson index or some other form? The selection of an inequality measure will depend upon the purpose of the exercise.

To give an example of a situation in which different inequality measure could be helpful, in joint work (Seth and Alkire 2013), one of us argues that inequality measures using deprivations can and perhaps should usefully be constructed to reflect absolute inequality, because this facilitates comparisons and also is coherent with a view that each deprivation is of intrinsic importance. Furthermore, the measure should be decomposable and should reflect both between and within group inequality levels. We select an inequality measure that could be called *variance,* because it uniquely satisfies a set of principles that are particularly important for policy, including subgroup consistency and decomposability within and between groups. In the case of variance, "its decomposable property allows the overall inequality to be decomposed into a total within-group and a between-group component…. Also, inequality remains same whether the poor are assessed in a deprivation space or in an attainment space." The inequality measure takes the form:

$$I(y) = \frac{\beta}{n}\sum_{i=1}^{n}[s_i(k) - \mu(s_i(k))]^2$$

In other words, the variance inequality measure is based on the censored vector of weighted deprivation scores, and reflects the average squared difference between person i's deprivation score and the mean deprivation score across the population, multiplied by some constant beta, which normalizes the inequality measure to lie between 0 and 1.

It is actually elementary to make other indices reflecting inequality among the multidimensionally poor. Above, we presented our measures with respect to an attainment vector a and associated poverty cutoff π. We showed how all well-known unidimensional measures could be applied to ordinal data that had been prepared using simple dichotomization, and that the interpretation of the entries in the attainment vector change from being the 'amount' of income to reflecting the 'breadth' of attainments across dimensions that people enjoy. In a parallel fashion, we now point out that it is elementary to generate any one of a set of possible measures of inequality among the poor, $I(a;\pi)$, using the attainment vector a censored by the associated poverty cutoff π. Recall that the attainment scores have some salience for multidimensional poverty, and potentially describe the variable attainments that people have in terms of breadth. Admittedly the attainment scores are crude, as they sum dichotomised variables, and the dichotomization omits any information on the levels of achievement or deprivation experienced. However this treatment respects the ordinal character of many categorical or nominal variables, and may be informative, particularly if the emphasis is inequality among the poor.

Using the distribution of (censored) attainment scores across the poor, we can create an inequality measure $I(a;\pi)$, much in the same way that traditional inequality measures such as Atkinson, Theil, or Gini are constructed. While it is clear that measures of inequality among the poor could draw upon the vector of censored attainment scores, it is also possible to construct legible measures of inequality that draw upon the distribution of *deprivation* scores, and indeed to use *uncensored* as well as censored versions of the score vectors. Measures constructed with these distributions will offer a window onto a certain type of multidimensional inequality – one that is oriented to the breadth of attainments people experience. This approach is quite different from other constructions of multidimensional inequality, but may be useful, particularly when data are ordinal.

7. Conclusion

We began by identifying three properties of the adjusted headcount ratio of Alkire and Foster (2011) that have made the poverty measure so useful in practice: subgroup decomposability, dimensional breakdown, and ordinality. We also discussed the two most common distribution sensitivity properties found in the literature, which require poverty not to fall in response to an increase in inequality among the poor, and noted that both are also satisfied by the adjusted headcount ratio. We explored alternatives to these weak forms of axioms and, given ordinality, settled upon an intuitive dimensional transfer property that requires poverty to rise in response to an inequality-increasing switch among the poor. Following a suggestion in Alkire and Foster (2011) we constructed a class of measures that satisfies this requirement that builds upon the adjusted headcount ratio and is related to the FGT indices. We devised a natural transformation from unidimensional to multidimensional poverty measures by converting a multidimensional distribution matrix into a single dimensional attainment count distribution and applied a unidimensional poverty measure. It was shown that if the unidimensional poverty measure satisfies the traditional monotonicity and transfer properties then the multidimensional poverty measure satisfies the dimensional monotonicity and dimensional transfer properties. The implication is that it is a straightforward exercise to construct examples of multidimensional measures that strictly reflect distributional considerations through the dimensional transfer axiom.

Each of these example measures, however, was found to violate dimensional breakdown, which then led us to prove an impossibility result identifying a fundamental conflict between the dimensional transfer and dimensional breakdown properties. We noted that there are many forms of inequality that are of interest to the poverty analyst, and that the form embodied in these distributional sensitivity properties is but one possibility. We showed how alternative analyses of inequality can be introduced while using a measure that satisfies the core property of dimensional breakdown. We illustrated these methods with an example the example of the Multidimensional Poverty Index (MPI) applied to Niger and broken down by region and by dimension. We also calculated several new multidimensional poverty measures obtained using the above transformation and analyze poverty by region, but not by dimension.

Several findings of this paper might be fruitfully developed in further research. The properties of new and existing multidimensional poverty measures clarified in terms of ordinality, dimensional breakdown, and dimensional transfer. The attainment count methodology could be used to generate and apply a large set of new multidimensional poverty measures. Further, the properties of these new measures, using the attainment count transformation, could be transparently discerned. Our distinction between the poverty cutoff used to identify the poor and the poverty standard used to aggregate information about the poor could be usefully applied to policy problems such as targeting.

By probing and clarifying the properties of the Adjusted Headcount Ratio, we conclude that that methodology continues to provide a neutral and policy-relevant foundation for empirical work. The policy value of being able to see the dimensional composition of poverty, to compare it across groups and analyse its change over time has been established. Analyses of inequality within and across subgroups of the poor can and should supplement this analysis, not replace it.

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