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Social Shock Sharing and Stochastic Dominance

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Abstract

In this paper, we provide intuitive justifications of normative restrictions based on the signs of fourth-order derivatives of utilities in the context of multidimensional welfare analysis. For this, we develop a new notion of welfare shock sharing. This allows us to derive new characterizations for symmetric and asymmetric conditions on the signs of fourth-order derivatives of utility functions. Then, we use these restrictions to derive new necessary and sufficient stochastic dominance criteria for multidimensional welfare comparisons, as well as equivalences in terms of multidimensional poverty measures.

Keywords: Multidimensional Welfare, Stochastic Dominance, Temperance, Risk Sharing.

JEL Codes: D3, D63, I31.

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1 Introduction

In this article, we deal with comparisons of inequality and social welfare across situations that can be described with several well-being attributes for each individual. For example, income, health and education are typically invoked as three relevant dimensions of individual well-being.

Multidimensional stochastic dominance criteria for social welfare and inequality analyses were put forward by Kolm (1977) and Atkinson and Bourguignon (1982). Their criteria were based on utility functions constrained by the signs of their partial derivatives up to the fourth order. However, the conditions involving fourth-order derivatives have not been recognized as being easy to interpret by most economists, as stated again in Atkinson (2003).

In typical stochastic dominance approaches of multidimensional welfare analysis, marginal utility functions, with respect to each attribute, are assumed to be identical across agents by invoking 'anonymity' axioms[1]. They are also generally supposed to be non-negative and non-increasing. However, these assumptions alone do not allow researchers to generate stochastic dominance criteria that would have sufficiently high discriminatory power to make them efficient guides for empirical economic policy. This is why economists have undertaken to reinforce these decision rules through incorporating hypotheses on signs of higher derivatives of utility functions. However, so far, it is fair to say that only criteria based on partial derivatives up to the third order, at most, are typically used[2]. This is because only limited normative justifications have been found to justify to push the analysis at higher orders. Despite this, Duclos, Sahn and Younger (2011) is an example of the use of Atkinson and Bourguignon conditions including fourth-order derivatives in an empirical context[3], which suggests that improved discriminatory

[1] In some cases, this can be justified by controlling explicitly for some differences in needs, as in Atkinson and Bourguignon (1987) for example.

[2] For example, Bazen and Moyes (2003), Gravel and Moyes (2012), Muller and Trannoy (2011, 2012).

[3] Some authors (e.g., Gravel and Mulhopadhay, 2009) propose empirical stochastic dominance ap-

power obtained this way can be attractive for empirical researchers. This induces us to pursue this research line, which is the first aim of this paper.

More generally, we investigate how to extend the expression of social solidarity in social welfare analysis, which is typically done through normative conditions on utility functions. For this, diverse axioms of monotonicity, transfers and compensation/substitution have already been well explored. To go further, we propose to introduce intuitions about social solidarity, which we state in terms of social sharing of individual shocks.

Social sharing notions for multidimensional welfare problems are not just motivated by climbing a ladder of conditions on utility derivatives. The variety of welfare shocks suffered by most households calls for a general setting that clearly specifies how social solidarities operate across the diverse dimensions of individual welfare. Shocks may affect health, income, physical safety, employment, family issues, environment, prices, etc. Considering heterogeneous shocks particularly relevant in some poor contexts, in which it has been found that many households cannot cope on their own with all these shocks[4]. In that case, traditional or modern institutions of social solidarity that can deal with the whole span of shocks are required. Public systems of social securities on the one hand, and families on the other hand, are examples of such institutions. These institutions assist households by sharing shocks, whether they are random or not, either through ex-post compensation devices such as cash transfers, or through ex-ante insurance or protection policies. Modern social security systems are becoming increasingly complex and sophisticated, dealing simultaneously with many different types of risks, handicaps, inequalities and other shocks, while this was always so for traditional sol-

plications based on even higher derivatives orders, for example by multiplying poverty headcounts or poverty gap indices at individual level. This is also typically the case in the one-dimensional literature using an infinite sequence of derivative conditions leading to stochastic dominance expressions (Kolm, 1976a,b, Fishburn and Willig, 1984). Again, normative justifications would be needed in order to better settle these practices.

[4] For example in Ethiopia in Kebede and Muller (2013).

idarity mechanisms. Then, investigating the extent to which multidimensional shocks should be socially shared, and how to account for them in welfare analysis, appears as relevant and even increasingly so.

In this paper, we discuss and develop new normative social notions of '*welfare shock sharing*'. These notions allow us to justify *separately* normative conditions in terms of variations in utilities faced to random or non-random shocks. We translate these conditions in terms of normatively meaningful signs of partial derivatives of utility functions (from the first to the fourth order).We also review other ways of justifying these signs conditions.

In a second stage, we consider the sets of utility functions that can be defined by using signs of partial derivatives up to the fourth-order. For each of these sets, we provide necessary and sufficient stochastic dominance theorems that allow the decision maker to compare multidimensional social welfare between two empirical situations. We also report equivalent results in the form of generalized poverty gap conditions.

The next section presents our setting. Section 3 discusses normative justifications based on the notion of welfare shock sharing. Section 4 reports our stochastic dominance results. Section 5 proposes an empirical application. Finally, we conclude in Section 6. Most of the proofs are in the appendix.

2 The Setting

We focus on the bivariate case, while most of what we shall discuss is valid for higher dimensions too. A typical example is the case where the first argument of the utility function is income and the second one is health. We consider a bivariate distribution of a random vector $(\tilde{x}, \tilde{y})$ over a set of random variable $\tilde{S}$. We assume that there random variable take values x and y on the rectangle $[0, a_1] \times [0, a_2] = A_1 \times A_2$, where a_1 and a_2 are in $\bar{\mathbb{R}}_+$. $\tilde{F}(\tilde{x}, \tilde{y})$ denotes the corresponding joint cumulative distribution function over the population of interest. The realisation of $\tilde{x}$ (respectively x) is denoted x (respectively y). Their joint cdf for a given $(\tilde{x}, \tilde{y})$ is denoted $F(x, y)$ and is assumed to be continuous, while $F_1(x_1)$ and $F_2(x_2)$ denote the respective marginal cdfs of x and y.

Let $W_{\tilde{F}}$ be an additively separable social welfare function, associated with cdf $\tilde{F}$

$$W_{\tilde{F}} := \int\limits_{\tilde{S}} \tilde{U}(\tilde{x}, \tilde{y}) d\tilde{F}(\tilde{x}, \tilde{y}),$$

where $\tilde{U}$ is a utility function from $\tilde{S}$ to $\mathbb{R}$. Often, in the social welfare literature, $\tilde{U}$ is replaced by a criterion of expected utility, which gives $W_{\tilde{F}} := \int\limits_{A_1 \times A_2} EU(\tilde{x}, \tilde{y}) d\tilde{F}(\tilde{x}, \tilde{y})$, where U is a cardinal von Neuman-Morgenstern utility function. When there is no randomness, one obtains $W_F := \int\limits_{A_1 \times A_2} U(x, y) dF(x, y)$, and this will be our starting point in the discussion. We shall examine how to consider social shocks, including some random shocks, starting from such situation.

We assume all the partial derivability properties needed to express our results. Moreover, in all the paper, we assume that all the considered integrals are bounded to avoid absurdities.[5]

Let $\triangle W_U := W_F - W_{F^*}$ be the change in social welfare between any two distributions

[5] If $a_1 = +\infty$ or $a_2 = +\infty$, then some integrals may not be defined for certain theoretical distributions,even with non random variables x_1 and x_2 such that $F(x_1, x_2)$ has heavy tails. This may also be the case for some integrals of some partial derivatives of utility arising in expansions. However, these cases are of little empirical relevance.

F and F^*. We obtain $\triangle W_U = \int\limits_{A_1 \times A_2} U(x,y)d\triangle F(x,y)$, where $\triangle F$ denotes $F - F^*$. Social welfare dominance is defined to correspond to the unanimity over a given set $\mathcal{U}$ of utility functions U.

Definition 1 *F dominates F^* for a family $\mathcal{U}$ of utility functions if and only if $\triangle W_U \geq 0$ for all utility functions U in $\mathcal{U}$. This is denoted $F D_{\mathcal{U}}\ F^*$.*

To be more specific about such dominance relationships, we need to define relevant sets $\mathcal{U}$ of utility functions. We proceed to do so in the next sub-sections, starting with some useful sets of functions generalising usual concavity notions.

2.1 A few definitions

We first introduce a few definitions for some generalised concave functions of (x, y) that we shall use later. Denuit, Lefèvre and Mesfioui (1999), Denuit and Mesfioui (2010) and Denuit, Eeckhoudt, Tsetling and Walker (2010) provide generator functions for these classes, which we use to derive stochastic dominance conditions in Section 4.

Definition 2 *Consider the functions of (x, y) from $A_1 \times A_2$ to $\mathbb{R}$. The $(s_1, s_2)-$increasing concave ($(s_1, s_2) - icv$) functions are all appropriately derivable functions g such that*

$(-1)^{k_1+k_2+1} \frac{\partial^{k_1+k_2}}{\partial x^{k_1} \partial y^{k_2}} g \geq 0,$

where $k_i = 0, .., s_i; i = 1, 2; s_1$ and s_2 are two non-negative integers and $1 \leq k_1 + k_2$. The corresponding classes of functions are denoted $\mathcal{U}_{(s_1,s_2)-icv}$.

The $s-$increasing directionally concave functions (s-idircv) are all functions g such that

$(-1)^{k_1+k_2+1} \frac{\partial^{k_1+k_2}}{\partial x^{k_1} \partial y^{k_2}} g \geq 0,$

where k_1 and k_2 are two non-negative integers and $1 \leq k_1 + k_2 \leq s$, where s is an integer larger than or equal to 2. The corresponding class of functions is denoted $U_{s-idircv}$.

Let $R_s = \{(r_1, r_2) \in N^2 |\ 1 \leq r_1 + r_2 = s\}$, which is used in the next theorem.

Theorem 1 *Let s be an integer greater of equal to n. Then,*

$$U_{s-idircv} = \bigcap_{(r_1,r_2)\in R_s} U_{(r_1,r_2)-icv}.$$

IIUThe s-idircv classes embody symmetrical assumptions that make them particularly liable to be characterised by asymptotic developments through symmetric derivations across all variables. In turn, these developments can be used to identify generator functions, which can then be mobilised to obtain necessary and sufficient conditions of stochastic dominance results.

2.2 The utility sets of interest

From now, in order to alleviate notations, we denote partial derivatives by using indices of attributes (1 and 2), repeated as many times as there is a derivation with respect to the attribute. For example, U_{1122} is $\frac{\partial^4}{\partial x^2 \partial y^2}U$.

Conditions on signs of partial derivatives were introduced progressively in the literature. For example, Levy and Paroush (1974) were the first ones, to our knowledge, to propose to use the condition $U_{12} \leq 0$. Atkinson and Bourguignon (1982) proposed various classes of utility functions. Their largest class is defined by functions satisfying $U_1, U_2 \geq 0, U_{12} \leq 0$, while their smallest class is defined by the same restrictions to which are added $U_{11} \leq 0, U_{22} \leq 0,\ U_{112} \geq 0, U_{221} \geq 0, U_{1122} \leq 0$. Other authors propose classes with intermediate sets of restrictions.[6]

In order to increase the power of the stochastic dominance tests, one may want to assume as many restrictions as possible. Let $\mathcal{U}$ be the class of increasing utility functions that satisfy the following signs for the partial derivatives.

[6] Moyes (1999), Bazen and Moyes (2003, Gravel and Moyes (2012), Muller and Trannoy (2011, 2012).

$$\begin{aligned}
\mathcal{U} &= \{U_1, U_2 \geq 0, U_{11} \leq 0, U_{22} \leq 0,\ U_{12} \leq 0, \\
U_{111} &\geq 0, U_{222} \geq 0, U_{112} \geq 0, U_{221} \geq 0, & (1) \\
U_{1111} &\leq 0, U_{2222} \leq 0, U_{1122} \leq 0, U_{1112} \leq 0, U_{1222} \leq 0\}. & (2)
\end{aligned}$$

This class involves a complete set of sign restrictions on partial derivatives up to the fourth order. Other conditions with opposite signs could also be considered, although they would yield rather counter-intuitive meanings (e.g., risk-loving or inequality-loving decision-maker with $U_{11} \geq 0$). We therefore only consider the most relevant signs for our analyses.

We now comment on these restrictions. The non-negativity of the first-order derivatives means that both attributes positively contribute to utility, or at least are not noxious to it. The non-positivity of the two direct second-order derivatives may be seen as expressing inequality aversion in social welfare settings. They correspond to the concavity of the utility in the directions of each attributes. However, they do not imply the global concavity of the utility in general. Besides, it is unclear whether global concavity would carry some relevant normative meaning.

IIUThe hypotheses $U_{12} \leq 0, U_{112} \geq 0$ and $U_{122} \geq 0$ can be justified as in Muller and Trannoy (2011, 2012) by invoking normative compensation arguments. Rather than treating all attributes as symmetrical, one of them is assumed to serve as a compensating variable to redress inequality with respect to certain needs. In that case, the first argument (e.g., income) is assumed to be useful for compensating possible destitution in the second argument (e.g., health). The more destitute in health an individual is, the higher the justification for compensating income transfers are. Moreover, with $U_{112} \geq 0$, such justification is all the more vindicated that potential transfer beneficiaries are poor. There are other possible justifications of these hypotheses. For example, $U_{12} \leq 0$ can be seen as embodying aversion for correlations between attributes, as in Tsui (1999). So, far justifications of signs of the fourth-order derivatives are missing in the literature.

Class $\mathcal{U}$ can be seen as corresponding to some 'maximum requirement' in terms of the signs of the partials since its definition cumulates a complete set of restrictions on signs up to the fourth order. However, other utility classes could be considered that would involve fourth-order partial derivatives, even though not all of the terms in the definition of $\mathcal{U}$. For example, the hypotheses in Atkinson and Bourguignon correspond to the class:

$\mathcal{U}^{--} = \{U_1, U_2 \geq 0, U_{11}, U_{22},\ U_{12} \leq 0, U_{121}, U_{212} \geq 0, U_{1122} \leq 0\}$, which may also be described as Class $(2,2) - icv$.

We also consider the following classes that involve fourth-order derivatives.

Class $(3,1) - icv$ corresponds to: $U_1, U_2 \geq 0$; $U_{11}, U_{12} \leq 0$; $U_{112}, U_{111} \geq 0$; $U_{1112} \leq 0$, which have never been considered jointly in the literature.

Class $(4,0) - icv$ corresponds to: $U_1 \geq 0$; $U_{11} \leq 0$; $U_{111} \geq 0$; $U_{1111} \leq 0$ and no condition on the other attribute. Classes $(1,3) - icv$ and $(0,4) - icv$ can be easily obtained by symmetry.

Our class $\mathcal{U}$ of main interest is also Class 4-idircv. It imposes symmetric restrictions on marginal variations in all directions. In order to justify normatively all these restrictions, we introduce new normative axioms in the next section, based on the new notion of 'welfare shock sharing'.

3 Normative Justifications

3.1 Welfare shock sharing

There are many ways of incorporating notions of shocks into welfare contexts. A few ones that come to mind are the specifications of: compensation for observed damages, vulnerability indices, drawing of social positions (e.g., anonymity axioms or ignorance veils), aversion to some welfare consequences of risks. Our approach is to specify shocks that affect individuals and that the social planner should consider while assessing social situations. In that sense, these shocks can be seen as 'social shocks'. Such shocks may happen randomly or not.

To fix ideas, let us consider a bivariate problem described by endowments (x, y) to individuals in a population. For example, x may be income and y health status, both considered as positive variables. Of course, any other welfare attributes could be considered if wished[7]. Let us further assume that society is only composed of two individuals, and let us examine the social planner's preferences for equity across individuals. For example, a planner who would be reluctant to see the same individual bearing all the shocks, would prefer the social situation $\{(x - c, y); (x, y - d)\}$, where the first individual has endowments $x - c$ in the first attribute and y in the second attribute, and the second individual has respective endowments equals to x and $y - d$, with $c, d > 0$ being fixed losses, to the social situation, or 'society', where the same individuals have the respective endowments $\{(x, y); (x - c, y - d)\}$, in which an individual would suffer all the losses. That is: in that case, the social planner prefers the situation where the allocation of shocks is 'shared' among individuals. However, there may be other types of shocks and other ways to share shocks among individuals.

More generally and precisely, we now state a few definitions of (social planner) preferences favouring welfare shock sharing. Let us start again with two individuals with the same bivariate endowments, respectively (x, y) and (x, y). Then, we consider the effect of diverse individual shocks from the point of view of the preferences of a social planner. Each of the following definitions may be seen as a plausible normative axiom in terms of some notions of shock sharing. The last definition involves four individuals.

Definition 3 *Let be any endowments $(x, y) \in R_+^2$. Let c and $d > 0$. Let ε be a centered numerical random variable and δ be a centered numerical random variable independent of ε.*

[7] As well as variables describing individual needs, although in the latter case the signs of derivatives we use in this paper would have to be changed accordingly, as in Bourguignon (1989) for example.

(i) *A social planner is said to be* welfare correlation averse *if* $x-c>0$ *and* $y-d>0$ *implies that the social planner prefers the society* $\{(x-c,y);(x,y-d)\}$ *to the society* $\{(x,y);(x-c,y-d)\}$. *That is: 'sharing fixed losses affecting different attributes is good'.*

(ii) *A social planner is said to be* welfare prudent *in* x *if* $x+\varepsilon>0$ *and* $x-c>0$ *implies that the planner prefers the society* $\{(x-c,y);(x+\varepsilon,y)\}$ *to the society* $\{(x-c+\varepsilon,y);(x,y)\}$. *That is: 'sharing a fixed loss and a centred risk affecting the same attribute is good'.*

(iii) *A social planner is said to be* welfare cross-prudent *in* x *if* $y+\delta>0$ *and* $x-c>0$ *implies that the planner prefers the society* $\{(x,y+\delta);(x-c,y)\}$ *to the society* $\{(x,y);(x-c,y+\delta)\}$. *That is: 'sharing a fixed loss and a centred risk affecting different attributes is good'.*

(iv) *A social planner is said to be* welfare temperate *in* x *if* $x+\varepsilon>0$, $x+\delta>0$ *and* $x+\delta+\varepsilon>0$ *implies that the planner prefers the society* $\{(x+\delta,y);(x+\varepsilon,y)\}$ *to the society* $\{(x,y);(x+\delta+\varepsilon,y)\}$. *That is: 'sharing centred risks affecting the same attribute is good'.*

(v) *A social planner is said to be* welfare cross-temperate *if* $x+\varepsilon>0$ *and* $y+\delta>0$ *implies that the planner prefers the society* $\{(x+\varepsilon,y);(x,y+\delta)\}$ *to the society* $\{(x,y);(x+\varepsilon,y+\delta)\}$. *That is: 'sharing centred risks affecting different attributes is good'.*

(vi) *A social planner is said to be* welfare-premium correlation averse *in* x, *if* $x+\varepsilon>0, x-c+\varepsilon>0$ *and* $y-d>0$ *implies that the planner prefers the society* $\{(x-c,y);(x,y-d);(x+\varepsilon,y);(x+\varepsilon-c,y-d)\}$ *to society* $\{(x,y);(x-c,y-d);(x+\varepsilon-c,y);(x+\varepsilon,y-d)\}$. *That is: 'Sharing fixed losses affecting different attributes is good, while less so under background risk in the first attribute'.*

In addition, the explicit definitions for monotonicity and inequality aversions, with respect to each attribute, could be explicitly stated and included in the list of definitions. However, we omit them as they are trivial. Symmetric definitions can of course also be obtained by substituting the roles of x and y, and we also omit them.

In the definition (vi), the society can be split into two subgroups of two individuals each. The welfare of the first two individuals can be compared across situations accordingly to welfare correlation aversion, which induces a preference for sharing losses. However, the last two individuals are ranked in the opposite ranking in this case with the same losses, the only difference being that they suffer in addition a centered random shock on the first attribute. It turns out that the total result of all these comparisons is deemed to be positive overall.

We shall show that under the expected utility hypothesis, the '*Welfare-Premium Correlation Aversion*' corresponds to the definition of welfare correlation aversion with the utility function in the definition (i) now replaced by the utility premium function, that is: $p^x(x, y, \varepsilon) = U(x, y) - EU(x + \varepsilon, y)$. Note that this premium notion corresponds to the comparison of the situation of two individuals whose individual welfare is assessed using their expected utility, instead of the comparison of random states in risk analysis. The premium is the amount of cardinal utility an individual would accept to pay for avoiding the be someone suffering the random risk as compared to someone not bearing it. For example, this comparison can be seen as performed by the individual themselves under a veil of ignorance to guarantee that individual idiosyncratic characteristics do not affect the decision. The utility premium is justified by a possible welfare loss from the random shock ε affecting the first attribute, under expected utility with concave VNM utility function. Definition (vi) can be seen as related to assessing the introduction of a background risk ε in the problem of correlation aversion to constant shocks.

Some of the stated definitions are akin to some notions used in risk analysis (prudence, temperance). That is why we use a similar vocabulary even though there are some differences between welfare and risk contexts. The first distinction is that we deal with social welfare comparisons instead of individual decisions under risk. The second one is that our notions are originally defined in terms of preferences over society situations, without invoking any representative function, for example utility functions, let alone expectations of von Neumann Morgenstern utility functions. Though, the particular application we examine is here in terms of expected utility. The third one is that the kind of risk apportionment that has been used for risk problems is sometimes formally different from our shock sharing formulae. The fourth one is that, for some notions (welfare cross-temperance, welfare premium correlation aversion), we have been unable to find any analog in the risk literature, even with broadened interpretation[8]. The fifth one is that the anonymity axioms typically used in the social welfare literature imply that the notions must be robust to some changes in the ranking or in the positions of the individuals. We express this by using set notations instead of n-uplet notations as it is typical in risk problems. The sixth one is that by invoking feelings of social justice, the axioms we propose seem to be much more likely to gather agreement than similar axioms defined in a context of risk apportionment, where they would look more arbitrary (e.g., temperance). The seventh one is that our definition of utility premium and social cost of risk involves the comparison of individuals, for example under a veil of ignorance, instead of the comparison of random states. The last one is that all attributes are supposed to be non-negative, since welfare attributes usually are, while it is not the case for random financial returns for example.

Note that some notions (i.e., $(ii), (iv)$) are independent of the presence of the other attribute, as long as this other attribute is kept fixed at a given level. Moreover, our notions of shock sharing do not generally depend on assuming specific levels of the endowments (x, y) since they are defined for any such endowments**IIU**. However, for each defined notion some endowment levels are excluded from consideration in order to impose that the ex-post endowments are positive and avoid absurdities. Indeed, many welfare variables have a natural lower bound, for example education that cannot be negative or consumption expenditure that cannot be below some subsistence minimum.

[8]Note however a draft paper by Crainich, Eeckhoudt and Courtois (2013) that was signaled to me by Pr. Eeckhoudt after he saw our paper submitted at the 2013 Ecineq conference.

Then, the corresponding shocks must be truncated[9]. Alternatively, some finite or infinite supports $[\underline{x}, \bar{x}] \times [\underline{y}, \bar{y}]$ could be used.

The results are valid for all $c, d, \varepsilon, \delta,$ that satisfy the domain constraints. However, only one given version of these parameters and shocks is necessary in the above definitions that are therefore weaker than if the properties were a priori imposed for all $c, d, \varepsilon, \delta$**.** Yet, little substantial outcome seem to be achievable from this distinction, unless there is an interest only in some given specific shocks.

As mentioned before, some of our new welfare shock sharing notions can be somewhat connected to some risk-apportionment notions in the literature. Eeckhoudt and Schlesinger (2006) introduce risk-apportionment techniques to characterise unidimensional prudence and temperance notions. Eeckhoudt, Rey and Schlesinger (2007) and Jokung (2011) extend these notions to bivariate settings. Eeckhoudt and Schlesinger (2006), and Eeckhoudt, Rey and Schlesinger (2007) characterize von Neuman-Morgenstern utility functions for expected utility criteria by using prudence, temperance, correlation aversion, cross-prudence and cross-temperance notions. However, in Eeckhoudt and Schlesinger (2006) and in Jokung (2011), 'high-order' risk preferences are constructed through a binary recurrence process over lotteries, which is clearly distinct from our approach, although they would coincide for low order notions such as risk aversion. Indeed, Eeckhoudt and Schlesinger, and Jokung define risk-apportionment notions from a sequence of risks, each defined recursively. In contrast, we directly define shock sharing notions without using a recurrent sequence of risks.

Note that there is no obvious reason why the expected utility criteria of individual welfare should necessarily be chosen as building blocks of our social welfare setting, even if some shocks are random. However, this special case should of course be included in most plausible settings and we stick to it in this paper. In general, individual risks can often be seen as a socially risky situations, and they can be incorporated in the setting

[9] One could also specify how dying individuals are deal with in the welfare setting, while it is not our interest in this paper.

of social welfare criteria. For example, a von Neumann-Morgenstern (VNM) ranking of socially risky situations that is consistent, in the Pareto sense, with individual VNM utilities, may be seen as resulting from comparing the sum of the individual's VNM expected utility functions (Harsanyi, 1955). Other aggregation theorems (Weymark, 1991 and 1993, Danan, Gajdos and Tallon, 2013) yield similar constructions, in particular for incomplete preferences. From now, we therefore follow this representation of aggregate social decision criteria based on sums of expected utilities.

Our first result of social welfare analysis is the following. In social welfare contexts such that the social evaluation function is additive in individual utility functions of possibly random attributes, we can characterise our axioms in the above definitions by signs of partial derivatives of the utility function, up to the fourth order, as shown in the next theorem.

Theorem 2 *Under the expected utility hypothesis representing individual welfare, we have;*

(a) Inequality Aversion is equivalent to $U_{11} \leq 0$*. An alternative interpretation is that of preferences for sharing fixed losses in the first attribute.*

(b) Welfare Correlation Aversion is equivalent to $U_{12} \leq 0$*.*

(c) Welfare Prudence in x *is equivalent to* $U_{111} \geq 0$*.*

(d) Welfare Temperance in x *is equivalent to* $U_{1111} \leq 0$*.*

(e) Welfare Cross-Prudence in x *is equivalent to* $U_{122} \geq 0$*.*

(f) Welfare Cross-Temperance is equivalent to $U_{1122} \leq 0$*.*

(g) Welfare-Premium Correlation Aversion in x *is equivalent to* $U_{1112} \leq 0$*.*

Obviously, similar properties can be obtained by substituting the roles of x and y, and we omit their explicit statements. The proofs are relegated in the appendix. They rely on the fact that, on an interval, corresponding finite variations and derivatives have the same sign when the sign of the derivatives is constant.

Results (b) and (c), in our specific example with income and health, imply that cross-prudence in income, $U_{122} \geq 0$, along to $U_{12} \leq 0$, can be seen as depicting a motive for compensation to alleviate inequity in health through income transfers in favour of ill persons, as exhibited in Muller and Trannoy (2012).

Justifying normative restrictions in social welfare analysis, by invoking Pigou-Dalton transfer-type arguments is often controversial when not all attributes can actually be transferred, for example health status. Using instead normative assumptions based on our shock sharing axioms diminishes this difficulty[10]. In any case, the concrete mechanisms or institutions through which the shocks could be actually shared is another matter, which is not dealt with in this article. We now briefly review other justifications of the above signs of the fourth derivatives of utility.

First, the signs of some utility derivatives may be related to variations in aversion to inequality. For example, $U_{1122} \leq 0$ is equivalent to U_{22} concave in x_1. In that case, decreasing income transfers increase aversion to health inequality (in terms of the concavity of U in x_2). This would correspond to original transfer axioms if one wished to attach a normative meaning to such change in health inequality aversion.**m.d. IIT**

Second, specific functional forms of utility functions can be invoked to justify some signs of utility derivatives. For example, if $U = V[\Phi(x_1) + \Psi(x_2)]$, where V, Φ and Ψ are some functions of type C^4, then we have the following results (The proofs are in the Appendix). If $\Phi' \geq 0, \Psi' \geq 0, V' \geq 0$ and $V'' \leq 0$, then $U_1, U_2 \geq 0$ and $U_{12} \leq 0$. If moreover $\Phi'' \leq 0$ and $\Psi'' \leq 0$, then $U_{11} \leq 0$ and $U_{22} \leq 0$ too. If moreover $V''' \geq 0, \Phi''' \geq 0$ and $\Psi''' \geq 0$, then, in addition, we obtain $U_{111}, U_{112}, U_{122}$ and $U_{222} \geq 0$. If moreover $V^{(4)} \leq 0$, then $U_{1122} \leq 0$. Finally, if furthermore $\Phi^{(4)} \leq 0$ and $\Psi^{(4)} \leq 0$, then we have $U_{1111} \leq 0, U_{2222} \leq 0, U_{1112} \leq 0$ and $U_{1222} \leq 0$. In that sense, the signs of the utility partial derivatives can here be justified by the signs of one-dimensional

[10] Shock sharing axioms are mathematically more general than transfer axioms for more than two attributes because there may be several equivalent doubly stochastic matrices, and some of them cannot be represented by transfers (Arnold, Marshall and Olkin, **pp??, 20??**). **voir si l argument est juste**

derivatives of the building-block functions, up to the fourth order. One-dimensional notions, such as monotonicity, inequality aversion, prudence and temperance applied to V, Φ and Ψ may be used as suggesting assumptions in that case.

This result is interesting in that it may help relaxing cardinality hypotheses by transforming utilities using arbitrary non-decreasing functions V, as long as they are fourth time differentiable in the points of interest and satisfy in these points $V'' \leq 0$, $V''' \geq 0$, $V^{(4)} \leq 0$.

Finally, the condition $U_{1111} \leq 0$ can be related to one-dimensional temperance or other notions already developed in the risk literature. For example, increasing outer inequality could naturally be defined as corresponding to negative fourth derivatives, as a generalization and translation of Menezes and Wang (2005) for increasing outer risk. Other notions that can be invoked to justify such sign are: Proper risk aversion (Pratt and Zeckauser, 1987), Decreasing absolute prudence (Kimball, 1993), Risk vulnerability (Gollier and Pratt, 1996).

In the next subsection, we introduce and discuss a novel normative condition based on fourth order partial derivatives.

3.2 A new notion of welfare shock sharing

The conditions $U_{1222} \leq 0$ and $U_{1112} \leq 0$ have been left out in the social welfare literature[11]. We now characterise them. Without loss of generality, let us consider $U_{1112} \leq 0$ for social preferences. As before, the characterization will rely on the fact that on an interval, finite variations and derivative have the same signs when the derivative sign is constant. We now spell out the proof fully in the text.

Let c be any fixed positive loss amount, and ε be any given centered random shock such that $x - c + \varepsilon > 0$ and $x - c > 0$, for all x. To fix ideas variable x may be considered to be income and y to be health. Let the *utility loss function* $w(x, y; c) =$

[11] See however footnote 8 for a recent investigation by Crainich et al. in the risk literature.

$U(x,y) - U(x-c,y)$, which describes the utility loss due to a fall in the first attribute. Let the Jensen's gap corresponding to function w: $v(x,y) = w(x,y;c) - Ew(x+\varepsilon, y; c)$.

Consider the condition $v_2(x,y) \leq 0$, which is akin to saying that the *utility premium* $p^x(x,y,\varepsilon) = U(x,y) - EU(x+\varepsilon,y)$ is subject to correlation aversion. Indeed, $v_2 = U_2(x,y) - U_2(x-c,y) - EU_2(x+\varepsilon,y) + EU_2(x+\varepsilon-c,y) = p_2^x(x,y,\varepsilon) - p_2^x(x-c,y,\varepsilon) \leq 0$, which is equivalent to $p_{12}^x \leq 0$. We now describe this condition in two equivalent ways: on the one hand, be examining the sign of a fourth-order partial derivative of utility; and, on the other hand, by comparing the total utility outcomes for two societies that respectively share or not these shocks.

Now, $v_2(x,y) = w_2(x,y;c) - Ew_2(x+\varepsilon,y;c) \leq 0$ if and only if w_2 is concave in x. Assuming derivability when needed, this condition is equivalent to $w_{112} \leq 0$, for any levels of the attributes, that is: $U_{1112} \leq 0$, since finite variations and derivatives have the same constant sign on an interval. Then, the condition ($U_{1112} \leq 0$) that we want to characterise is equivalent to $v_2(x,y) \leq 0$, for all x, y.

The next step consists in noting that we have $v_2(x,y) \leq 0$ for all x, y, c, ε such that $y > 0, x - c + \varepsilon > 0$ and $x - c > 0$ if and only if

$w(x,y;c) - Ew(x+\varepsilon,y;c) - w(x,y-d;c) + Ew(x+\varepsilon,y-d;c) \leq 0$, for all such x, y, c, ε and d, through replacing v and finite variation approximation. This yields, by replacing w: $U(x,y) - U(x-c,y) - EU(x+\varepsilon,y) + EU(x+\varepsilon-c,y)$

$-U(x,y-d) + U(x-c,y-d) + EU(x+\varepsilon,y-d) - EU(x+\varepsilon-c,y-d) \leq 0$, for all such x, y, c, ε and d. By reordering terms, this condition can be rewritten as

$U(x-c,y) + U(x,y-d) + EU(x+\varepsilon,y) + EU(x+\varepsilon-c,y-d)$

$\geq U(x,y) + U(x-c,y-d) + EU(x+\varepsilon-c,y) + EU(x+\varepsilon,y-d)$.

That is, providing one uses the expected utility as a valid welfare criterion, the four-individuals society $\{(x-c,y);(x,y-d);(x+\varepsilon,y);(x+\varepsilon-c,y-d)\}$ is preferred to the four-individuals society $\{(x,y);(x-c,y-d);(x+\varepsilon-c,y);(x+\varepsilon,y-d)\}$.

The smoothing of the 'background risk' ε reduces the concavity in x of the expected utility function $EU(.+\varepsilon,y)$, as compared to the original utility function $EU(.,y)$. This makes inequality issues looking less severe for individuals subjected to the background risk. This is the separation of the considerations pertaining to risk (through the expectation of utility and the corresponding utility premium) and to inequality (through welfare shock sharing) that allows us to elicit this feature. We now gather our results in the next theorem.

Theorem 3 *For all* x, y, c, d, ε *such that* $x - c > 0, y - d > 0, x + \varepsilon > 0, x - c + \varepsilon > 0$, *providing the expected utility criterion is used as the individual welfare measure, Condition* $U_{1112} \leq 0$ *is equivalent to:*

> *The four-individuals society*
> $\{(x - c, y); (x, y - d); (x + \varepsilon, y); (x - c + \varepsilon, y - d)\}$
> *is weakly socially preferred to the four-individuals society*
> $\{(x, y); (x - c, y - d); (x - c + \varepsilon, y); (x + \varepsilon, y - d)\}$.

Of course, with similar proofs than above, we can obtain a similar normative justification for the symmetrical condition: $U_{1222} \leq 0$. We have then provided a rigorous characterisation of Condition $U_{1112} \leq 0$. However, some readers may think that four-individuals societies may be harder to grasp intuitively than two-individuals societies. So, we now propose a two-individuals society characterisation.

The result in Theorem 3 could be interpreted as "sharing fixed losses affecting different attributes is good, while less so under background risk on the first attribute". Indeed, the situation of the first couple of individuals can be socially assessed as better in the first society than in the second society by invoking correlation aversion, as above in Theorem 2, i.e. $\{(x - c, y); (x, y - d)\} \succ \{(x, y); (x - c, y - d)\}$. On the contrary, the situation of the second couple of individuals corresponds to increased correlation of losses, while in presence of a background risk ε. Under correlation aversion, this would yield $\{(x + \varepsilon, y); (x - c + \varepsilon, y - d)\} \prec \{(x - c + \varepsilon, y); (x + \varepsilon, y - d)\}$. However, the background risk reduces the sensitivity of the social planner to inequality through expectation smoothing. This is why, on the whole, the planner can be assumed to consider that the degradation of the situation of the second subgroup of individuals is more than compensated by the improvement of the situation of the first subgroup of individuals. Returning to the utility premium, we have equivalently: $p^x(x-c, y, \varepsilon) + p^x(x, y-d, \varepsilon)$ is preferred to $p^x(x, y, \varepsilon) + p^x(x-c, y-d, \varepsilon)$. In that sense, the utility premium function embodies the potential social compensations of the risks across the two subgroups, and allows the planner to focus on correlation aversion from non-random losses.

Note that the above intuitive reasoning also suggests that it makes sense to assume $U_{12} \leq 0$ when assuming $U_{1112} \leq 0$ or $U_{1222} \leq 0$. Indeed, one of the proposed interpretation makes explicit use of the assumption of correlation aversion. Let us now turn to the stochastic dominance results that can be reached by assuming these new normative justifications of signs for the fourth order derivatives of utility.

4 Stochastic Dominance Results

We first need to define a few stochastic integrals that will be used to state our results.

4.1 Stochastic integrals

Definition 4 *Let*

$$H_x(x) = \int_0^x F_x(s)ds,\ L_x(x) = \int_0^x \int_0^t F_x(s)dsdt \text{ and } M_x(x) = \int_0^x \int_0^u \int_0^t F_x(s)dsdtdu,$$

$$H(x,y) = \int_0^x \int_0^y F(s,t)dsdt \text{ and } H_x(x;y) = \int_0^x F(s,y)ds,$$

$$L_x(x;y) = \int_0^x \int_0^s F(u,y)duds \text{ and } M_x(x;y) = \int_0^x \int_0^s \int_0^u F(t,y)dtduds,$$

and similar notations by substituting the role of x and y, respectively associated with indices that we also denote x and y.

In these definitions, the letter H indicates that the joint cdf F is integrated once with respect to a variable. The index variable is denoted by a subscript. The semi-colon in H_x, L_x and M_x indicates that the variable on the left-hand-side of the semi-colon is used for integration more times than the variable at the right-hand-side. The gap for these stochastic integrals between two distributions F and F^* is also denoted by using the operator $\triangle$, as in Section 2.

4.2 A few stochastic dominance results

In order to derive our stochastic dominance theorems, we shall avail of recent results on multidimensional stochastic orderings[12]. A first result is that the class $\mathcal{U}^{--}$ in Atkinson and Bourguignon may now become legitimately available to empirical researchers, as it has now clear normative interpretation by invoking the property of welfare cross-temperance. The result obtained in Atkinson and Bourguignon (1982) is the following and corresponds to classes of utility functions that are (2, 2)-increasing concave.[13]

Theorem 4 *(Atkinson and Bourguignon): For $(2,2)-icv$, that is: $U_1, U_2 \geq 0$; $U_{11}, U_{12}, U_{22} \leq 0$; $U_{112}, U_{221} \geq 0$; $U_{1122} \leq 0$, the following conditions are necessary and sufficient for stochastic dominance for continuous distributions.*

[12] For example, Theorem 7(i) in Denuit, Eeckhoudt, Tsetlin and Walker (2010).

[13] Note that in Atkinson and Bourguignon (1982), only the proof of the sufficient condition is given. The necessary conditions is omitted on the ground that it is an obvious generalisation of the unidimensional case. As a matter of fact, it is not and instead obtaining necessary conditions are often the main difficulty in such proof of necessary and sufficient results. More details on a proof of the necessary condition can be found in Atkinson and Bourguignon (1987) by going through the analogy of needs problems. However, still there the convergence of the proposed generators to the complete class of functions is presented as obvious, while it may be seen as a crucial difficulty in this kind of proof. In contrast, here the proof is a direct consequence of the knowledge of the generators of the class of $(2,2)-icv$ functions.

Regarder de pres cette difference dans AB87, et dans le papier de Maurin et al. qui les cite je crois

(a) $\triangle H(x, y) \leq 0$*, for all* x, y*.*

(b) $\triangle H_y(y) \leq 0$*, for all* y*.*

(c) $\triangle H_x(x) \leq 0$*, for all* x*.*

The conditions on the first-order derivatives indicate monotonicity with respect to the two attributes. The condition $U_{11} \leq 0$ (respectively $U_{22} \leq 0$) can be interpreted as stating some aversion for income (respectively for health) inequality, or alternatively in our interpretation, preference for income (respectively health) shock sharing. $U_{12} \leq 0$ describes welfare correlation aversion. $U_{111} \geq 0$ (respectively $U_{222} \geq 0$) is associated with welfare prudence in income (respectively in health), while $U_{112} \geq 0$ (respectively $U_{122} \geq 0$) means welfare cross-prudence in health (respectively in income). Finally, $U_{1112} \leq 0$ (respectively $U_{1222} \leq 0$) is equivalent to welfare-premium correlation aversion in income (respectively in health).

In contrast with the preceding class of utility functions, already examined by Atkinson and Bourguignon, the following class has never been studied to our knowledge. It satisfies the property of welfare-premium correlation aversion in income. However, it does not include conditions involving derivations with respect to the second argument more than once. Such setting may be appropriate for welfare problems involving an ordinal second attribute as in Bazen and Moyes (2003), and Gravel and Moyes (2012). We obtain.

Theorem 5 *For* $(3, 1)-icv$*, that is:* $U_1, U_2 \geq 0$*;* $U_{11}, U_{12} \leq 0$*;* $U_{112}, U_{111} \geq 0$*;* $U_{1112} \leq 0$*, the following conditions are necessary and sufficient for stochastic dominance.*

(a) $\triangle L_x(x; y) \leq 0$*, for all* x, y*.*

(b) $\triangle H_x(a_1; y) \leq 0$*, for all* y*.*

(c) $\triangle F_y(y) \leq 0$*, for all* y*.*

Condition (c) corresponds to first order stochastic dominance on the second attribute, often a quite demanding condition. This reflects that only first order derivations with respect to the second attribute have been used in the definition of the utility function class. The second condition is a mixed stochastic dominance term where the joint cdf is cumulated with respect to the first attribute, up to the corresponding upper bound. In particular, at the bound $y = a_2$, it implies $\triangle H_x(a_1) \leq 0$, which can be seen as a negative difference in a specific inequality measure in the first attribute between the two situations to compare.

When the marginal distributions of the second attribute are fixed, Condition (b) corresponds to the sequential generalized Lorenz criterion. In the general case, it can be expressed using Projected Generalized Lorenz tests, as shown in Muller and Trannoy (2012).

Finally, Condition (a) involves again a mixed stochastic dominance term, where this time the joint cdf is cumulated twice with respect to the first attribute, and for any level of the two attributes. At the bound $y = a_2$, it implies $\triangle L_x(x) \leq 0$, which corresponds to the well-known third-order one-dimensional stochastic dominance term.

The case of $(1,3) - icv$ is obviously symmetric. The next theorem corresponds to well-known results of one-dimensional fourth-order stochastic dominance.

Theorem 6 *For $4 - icv$, that is: $U_1 \geq 0, U_{11} \leq 0, U_{111} \geq 0; U_{1111} \leq 0$, the following conditions are necessary and sufficient for stochastic dominance.*

(a) $\triangle M_x(x) \leq 0$, *for all* x.

(b) $\triangle L_x(a_1) \leq 0$.

(c) $\triangle H_x(a_1) \leq 0$.

Beyond being a reminder, this theorem points out that the second attribute can be neglected in the analysis, as long as the imposed normative conditions do not involve utility derivatives with respect to this attribute. The reason why there is no first-order condition of the type $\triangle F_x(a_1) \leq 0$ in this sequence is that $F_x(a_1) = 1$ for the two distributions to compare, then the difference cancels out. A similar proposition could of course be stated with the other attribute y. It is finally possible to deal with the class of fourth order increasing directionally concave functions, which we do in the next theorem.

Theorem 7 *For* $4-$*idircv, that is:* $U_1, U_2 \geq 0$*;* $U_{11}, U_{12}, U_{22} \leq 0$*;* $U_{112}, U_{221}, U_{111}, U_{222} \geq 0; U_{1122}, U_{1112}, U_{1222}, U_{1111}, U_{2222} \leq 0$*, we have:*

(α)The 4-idircv class has a set of generators that is the intersection of the sets of generators of the (s_1, s_2)-icv functions with (s_1, s_2)$\in \{(2,2),(3,1),(1,3),(4,0),(0,4)\}$.

(β) Let be the change in variable from the algebraic form to the trigonometric form of complex numbers: $z = x + iy = \rho e^{i\theta}$ *with* $\rho = \sqrt{x^2 + y^2}$ *and* $\theta = Arg(z)$*, where* $x, y \in R, \rho \in R+$ *and* $\theta \in [0, \pi/2]$ *in the case* $a_1 = a_2 = +\infty$*, so as to impose the restrictions* $x \geq 0$ *and* $y \geq 0$*. Then,*

4-idircv in (x,y) is equivalent to 4-icv in ρ.

(γ) The necessary and sufficient conditions of stochastic dominance for the 4-idircv class are, in the case $a_1 = a_2 = +\infty$*:*

(a) $\triangle M_\rho(\rho) \leq 0$*, for all* ρ*.*

(b) $\triangle L_\rho(+\infty) \leq 0$*.*

(c) $\triangle H_\rho(+\infty) \leq 0$*.*

The conditions for other levels of a_1 *or* a_2 *correspond to an appropriate bound* a_ρ *in the expressions (b) and (c).*

(δ) The generators of the $4 - idircv$ class are the functions of x and y defined by: $\left(c - \sqrt{x^2 + y^2}\right)_+^{k-1}$, for all $c \in [0, a_\rho]$, if $k = 4$ and $c = a_\rho$ if $k = 1, 2, 3$, with $(z)_+ \equiv \max\{z, 0\}$.

Note that there is no problem of incompatibility of domain definitions when changing in variables between (x, y) and (ρ, θ), even if some of these variables are bounded. The conditions can be trivially adjusted by defining the relevant bounds of the joint domain in both representations.

Again, as in fourth-order one-dimensional stochastic dominance, we have a fourth-order term in condition (a) of (γ), although this time in terms of the modulus ρ. We can therefore hope for substantial gain in discriminatory power of empirical tests as compared with typical applications limited to second order stochastic dominance.

Results (α) makes the link between the generators of two kinds of classes of interest. However, this intersection property cannot be easily exploited to make explicit the generators of the 4-idircv class. This is because the generators of each $(r_1, r_2) - icv$ class are infinitely many since they depend on parameters that can take an infinite number of values. Furthermore, specifying completely the intersection of these generator sets through equations does not seem to lead to any tractable system to solve. However, Result (γ) provides the solution to this question by indicating that the generators of the 4-idircv class are simply the generators of the 4-icv class, which are known, while applied to the modulus variable. In the next subsection, we convert our stochastic dominance results into poverty orderings.

4.3 Poverty Orderings

Foster and Shorrocks (1988) showed that unidimensional stochastic dominance tests can be seen to be equivalent to some one-dimensional poverty orderings.

Definition 5 *The FGT Poverty measure of order α is:*

$P^{\alpha}(F, z) = \frac{1}{z^{\alpha}} \int_0^{F(z)} (z - F^{-1}(p))\, dp,$

where F is the cdf of incomes and z is the poverty line. The parameter $\alpha \geq 0$ is typically chosen equal to 0 (head-count index), 1 (poverty gap) or 2 (poverty severity index).

The range of poverty lines is denoted Z.

The poverty ordering $P^{\alpha}(Z)$ are defined as follows for two income distributions F and G: F $P^{\alpha}(Z)$ G if and only if

$P^\alpha(F,z) \leq P^\alpha(G,z)$ for all $z \in Z$, and $P^\alpha(F,z) < P^\alpha(G,z)$ for at least a $z \in Z$.

Cumulative integrals of the cdf F are defined recursively as follows: $F_1 \equiv F$ and $F_\alpha(s) \equiv \int_0^s F_\alpha(t)dt, \alpha \geq 2$.

With these notations, Foster and Shorrocks have shown that: $z^{\alpha-1}P^\alpha(F,z) = \int_0^s (z-y)^{\alpha-1}dF(y) = (\alpha-1)!F_\alpha(z)$. Thus, there is equivalence between the poverty ordering $P^\alpha(Z)$ and the α^{th} order stochastic dominance ordering. In particular, for all $\alpha \leq \beta$, $F\ P^\alpha(Z)\ G$ implies $F\ P^\beta(Z)\ G$. They also point out that $F\ P^2(Z)\ G$ is equivalent to F Generalised Lorenz dominating G. Equipped with these results, let us consider successively our classes of utility function of interest.

4.3.1 4-icv

This is the class generating the classical results of fourth-order stochastic dominance.

Therefore, using Foster and Shorrocks results, we know that this dominance ordering is equivalent to the poverty ordering $P^4(Z)$. Although this ordering has not been typically used in the one-dimensional stochastic dominance literature, we have now provided a shock sharing motivation to use it.

4.3.2 4-idircv

For the 4-idircv case, we can draw on Foster and Shorrocks classical results to state that the corresponding stochastic dominance ordering is equivalent to the poverty dominance ordering $P^4(Z_\rho)$ calculated for the $\blacksquare$ modulus variable, and for poverty lines $z_\rho \in Z_\rho$ defined in terms of this variable, where Z_ρ is the corresponding range of such poverty lines. This is a direct consequence of Theorem 7.

4.3.3 (3,1)-icv

The (3,1)-icv case necessitates to introduce notations for bivariate poverty indices. Let z_i be an absolute poverty line for the i^{th} attribute, $i = 1, 2$.

Definition 6 *The joint Poverty measure of order* (k_1, k_2)*, for the population deprived in* x *below a level* z_1 *and deprived in* y *below a level* z_2*, is:*

$$P_{k_1,k_2}(x, y; z_1, z_2) = \int\limits_{[0,z_2]}\int\limits_{[0,z_1]} (z_1 - x)^{k_1-1} (z_2 - y)^{k_2-1} \, dF(x, y).$$

Referring to the proof of Theorem 5, and translating to our setting, we have $F_{x,y} \succ_{(3,1)-icv} F^*_{x,y}$ if and only if

$$\begin{aligned} P_{k_1,k_2}(x_1, y_1; z_x, z_y) &= \int\limits_{[0,z_y]}\int\limits_{[0,z_x]} (z_x - x_1)^{k_1-1} (z_y - y_1)^{k_2-1} \, dF_{x_1,y_1}(x_1, y_1) \\ &\leq \int\limits_{[0,z_y]}\int\limits_{[0,z_x]} (z_x - x_2)^{k_1-1} (z_y - y_2)^{k_2-1} \, dF_{x_2,y_2}(x_2, y_2) = P_{k_1,k_2}(x_2, y_2; z_x, z_y), \end{aligned}$$

for all $z_x \in [\underline{x}, \bar{x}]$ if $k_1 = 3$ and $z_x = \bar{x}$ if $k_1 = 1, 2$ and for all $z_y \in [\underline{y}, \bar{y}]$ if $k_2 = 3$, and $z_y = \bar{y}$ if $k_2 = 1, 2$.

4.3.4 (2,2)-icv

we have $F_{x,y} \succ_{(2,2)-icv} F^*_{x,y}$ if and only if

$$\begin{aligned} P_{k_1,k_2}(x_1, y_1; z_x, z_y) &= \int\limits_{[0,z_y]}\int\limits_{[0,z_x]} (z_x - x_1)^{k_1-1} (z_y - y_1)^{k_2-1} \, dF_{x_1,y_1}(x_1, y_1) \\ &\leq \int\limits_{[0,z_y]}\int\limits_{[0,z_x]} (z_x - x_2)^{k_1-1} (z_y - y_2)^{k_2-1} \, dF_{x_2,y_2}(x_2, y_2) = P_{k_1,k_2}(x_2, y_2; z_x, z_y), \end{aligned}$$

for all $z_x \in [\underline{x}, \bar{x}]$ if $k_1 = 2$ and $z_x = \bar{x}$ if $k_1 = 1$, and for all $z_y \in [\underline{y}, \bar{y}]$ if $k_2 = 2$, and $z_y = \bar{y}$ if $k_2 = 1$.

4.3.5 Generalised Lorenz results

to do

5 Empirical Application

to do

6 Conclusion

to do

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Appendix

Proof of the sign conditions for the functional form $U = V[\Phi(x) + \Psi(y)]$:

The signs of the derivatives are deduced from the hypotheses and the following identities.

$U_1 = V'\Phi'$ and $U_2 = V'\Psi'$; $U_{12} = V''\Phi'\Psi'$.

$U_{11} = V''(\Phi')^2 + V'\Phi''$ and $U_{22} = V''(\Psi')^2 + V'\Psi''$.

$U_{111} = V'''(\Phi')^3 + 3V''\Phi'\Phi'' + V'\Phi'''$.

$U_{222} = V'''(\Psi')^3 + 3V''\Psi'\Psi'' + V'\Psi'''$.

$U_{112} = V'''(\Phi')^2\Psi' + V''\Phi''\Psi'$.

$U_{122} = V'''(\Psi')^2\Phi' + V''\Phi'\Psi''$.

$U_{1122} = V^{(4)}(\Phi'\Psi')^2 + V'''[(\Phi')^2\Psi'' + (\Psi')^2\Phi''] + V''\Phi''\Psi''$.

$U_{1111} = V^{(4)}(\Phi')^{(4)} + 6V'''(\Phi')^2\Phi'' + 3V''(\Phi'')^2 + 4V''\Phi'''\Phi' + V'\Phi^{(4)}$.

$U_{2222} = V^{(4)}(\Psi')^{(4)} + 6V'''(\Psi')^2\Psi'' + 3V''(\Psi'')^2 + 4V''\Psi'''\Psi' + V'\Psi^{(4)}$.

$U_{1112} = V^{(4)}(\Phi')^3\Psi' + 3V'''\Phi'\Phi''\Psi' + V''\Phi'''\Psi'$.

$U_{1222} = V^{(4)}(\Psi')^3\Phi' + 3V'''\Psi'\Psi''\Phi' + V''\Psi'''\Phi'$.

Proof of Theorem 1: Let $s \geq n$. Recall $R_s = \{(r_1, r_2) \in N^2 | \; 1 \leq r_1 + r_2 = s\}$. Any function $g \in \bigcap_{(r_1,r_2)\in R_s} \mathcal{U}_{(r_1,r_2)-icv}$ is such that $(-1)^{k_1+k_2+1} \frac{\partial^{k_1+k_2}}{\partial x^{k_1} \partial y^{k_2}} g \geq 0$, which we denote property $P(k_1, k_2)$, and is satisfied for $k_1 = 0, ..., r_1; k_2 = 0, ..., r_2; k_1 + k_2 \geq 1$; for any $1 \leq r_1 + r_2 = s$.

In particular, we can now show that for any $g \in \bigcap_{(r_1,r_2)\in R_s} \mathcal{U}_{(r_1,r_2)-icv}$, we have also $P(k_1, k_2)$ true for any (k_1, k_2) such that $1 \leq k_1 + k_2 \leq s$. Indeed, there exist some $(r_1, r_2) \in R_s$ such that the k_1 in the range $0, .., r_1$, and the k_2 is in the range $0, .., r_2$, with $k_1 + k_2 \geq 1$. We have the sum $k_1 + k_2 \leq r_1 + r_2 = s$, by construction. An example of such (r_1, r_2) is $r_1 = k_1$ and $r_2 = s - r_1$. Therefore, $g \in \mathcal{U}_{s-idircv}$. We have then shown $\mathcal{U}_{s-idircv} \supset \bigcap_{(r_1,r_2)\in R_s} \mathcal{U}_{(r_1,r_2)-icv}$.

Reciprocally, let $g \in \mathcal{U}_{s-idircv}$ and any given $(r_1, r_2) \in R_s$. We want to show that $g \in \mathcal{U}_{(r_1,r_2)-icv}$. We know that $P(k_1, k_2)$, for any k_1 and k_2 non-negative integers such that $1 \leq k_1 + k_2 \leq s$. In particular, this is satisfied for all (k_1, k_2) such that $k_1 \leq r_1, k_2 \leq r_2$ since in that case $k_1 + k_2 \leq r_1 + r_2 = s$. Therefore, $g \in \mathcal{U}_{(r_1,r_2)-icv}$. Since this reasoning can apply for any $(r_1, r_2) \in R_s$, this implies $\mathcal{U}_{s-idircv} \subset \bigcap_{(r_1,r_2)\in R_s} \mathcal{U}_{(r_1,r_2)-icv}$. QED.

Proof of Theorem 2:

(a) For the condition $U_{12} \leq 0$, we start from $v(x, y) \equiv U_{12}(x, y)$ as an ancillary function. Then, we consider finite variations as approximations of the partial derivatives of U embodied in v. This is relevant here because on the whole considered domain, the fixed sign of these finite variations will also be the sign of the corresponding derivatives. Let $c > 0$ and $d > 0$ be any fixed constants such that $x - c > 0$ and $y - d > 0$. First, $U(x, y) - U(x - c, y)$ approximates U_1. Then, $U(x, y) - U(x - c, y) - U(x, y - d) + U(x - c, y - d)$ approximates U_{12}. As a result, $U_{12} \leq 0$ over the whole domain is equivalent to $U(x, y) + U(x - c, y - d) \leq U(x, y - d) + U(x - c, y)$ over the whole domain. Then, provided we assume a social welfare criterion that is additive in utility functions, e.g. utilitarianism, we have: society $\{(x - c, y); (x, y - d)\}$ is weakly preferred to society $\{(x, y); (x - c, y - d)\}$. Sharing among individuals shocks that are fixed losses is a social improvement even if the shocks affect different attributes.

(b) Starting instead from the condition $U_{11} \leq 0$, and using the same approximation method with $x-c-d>0$, we obtain $U(x,y)+U(x-c-d,y) \leq U(x-c,y)+U(x-d,y)$. With social welfare criteria that are additive in utilities, society $\{(x-c,y);(x-d,y)\}$ is weakly preferred to society $\{(x,y);(x-c-d,y)\}$. Sharing shocks that are fixed losses affecting the same attribute among individuals is a social improvement.

Note that this interpretation of shock sharing does not seem to have appeared so far directly in the welfare literature as an axiom. Indeed, this literature rather invokes inequality aversion motives.

Of course, $U_{22} \leq 0$ is liable to the same type of interpretation for the second attribute. Note however, that it is quite possible that $U_{11} \leq 0$ holds and not $U_{22} \leq 0$ (or the opposite) because the two attribute have distinct normative roles. For example, one could imagine a society prone to redistribute income while ignoring health differences for social policies.

voir ci dessous comment reordonner les conditions

(e) Let us now turn to the condition $U_{112} \geq 0$. Let ε be any centered shock and d any positive constant such that $x+\varepsilon>0$ and $y-d>0$. Define the utility premium function by $v(x,y) = p^x(x,y,\varepsilon) = U(x,y) - EU(x+\varepsilon,y)$. By deriving once with respect to the second attribute, we obtain $v_2(x,y) = U_2(x,y) - EU_2(x+\varepsilon,y)$. We now impose the following fixed sign over the whole domain: $v_2 \leq 0$. On the one hand, using Jensen's inequality with respect to the first attribute, this condition is equivalent to U_2 convex in x_1, which is equivalent to $U_{112} \geq 0$, the condition we are studying. On the other hand, $v_2 \leq 0$ over the whole domain is equivalent to $U(x,y) - EU(x+\varepsilon,y) - U(x,y-d) + EU(x+\varepsilon,y-d) \leq 0$, through finite variation approximation. Rearranging yields $U(x,y) + EU(x+\varepsilon,y-d) \leq U(x,y-d) + EU(x+\varepsilon,y)$, which implies that $\{(x,y-d);(x+\varepsilon,y)\}$ is weakly preferred to $\{(x,y);(x+\varepsilon,y-d)\}$. If one shock is a fixed loss and the other is a random centered shock on the other attribute, sharing them among individuals improves social welfare. Of course, the case $U_{122} \geq 0$ can be dealt with similarly.

(c) We now consider the condition $U_{111} \geq 0$. Using the same reasoning as just before, while allocating the fixed loss to the first attribute instead, we obtain

$U(x, y) + EU(x + \varepsilon - c, y) \leq U(x - c, y) + EU(x + \varepsilon, y)$, with c any positive constant and ε any centered shock such that $x + \varepsilon - c > 0, x - c > 0$ and $x + \varepsilon > 0$. Society $\{(x - c, y); (x + \varepsilon, y)\}$ is weakly preferred to $\{(x, y); (x + \varepsilon - c, y)\}$.

Again, we find an interpretation in terms of shock sharing of a fixed loss shock and a random shock on the same attribute between two individuals, which leads to social welfare improvement. Note that even in the risk context, such interpretation does not seem to have emerged from the literature. The case $U_{222} \geq 0$ is similar.

(f) The condition $U_{1122} \leq 0$ is treated by starting again with the utility premium function $v(x, y) = p^x(x, y, \varepsilon) = U(x, y) - EU(x + \varepsilon)$, with ε any centered shock such that $x + \varepsilon > 0$. However, we now derive twice with respect to the second argument to obtain $v_{22}(x, y) = U_{22}(x, y) - EU_{22}(x + \varepsilon)$. In these conditions, $v_{22}(x, y) \geq 0$ is equivalent to U_{22} concave in x_1, that is: $U_{1122} \leq 0$. On the other hand, $v_{22}(x, y) \geq 0$ can be characterized by the Jensen's inequality with respect to the second argument, which is applied to the utility premium function: $v(x, y) - Ev(x, y + \delta) \leq 0$, where δ is a random centered shock independent of ε. By replacing the definition of v, we get $U(x, y) - EU(x + \varepsilon, y) - EU(x, y + \delta) + EU(x + \varepsilon, y + \delta) \leq 0$. Rearranging leads to $U(x, y) + EU(x + \varepsilon, y + \delta) \leq EU(x + \varepsilon, y) + EU(x, y + \delta)$. This yields: Society $\{(x, y + \delta); (x + \varepsilon, y)\}$ is weakly preferred to Society $\{(x, y); (x + \varepsilon, y + \delta)\}$. In that case, that is the sharing of two random shocks, each on a different attribute, between individuals that enhances social welfare.

(d) For the condition $U_{1111} \leq 0$, the proof of the previous case can be replicated by allocating the random centered shock δ to the first attribute instead, still with δ and ε any centered random shocks mutually independent such that $x + \varepsilon > 0, x + \varepsilon + \delta > 0$ and $x + \delta > 0$. This leads to $U(x, y) + EU(x + \varepsilon + \delta, y) \leq EU(x + \varepsilon, y) + EU(x + \delta, y)$, which can be interpreted in terms of shock sharing preferences as before, with here the random shocks affecting the same attribute. The case $U_{2222} \leq 0$ is similar.

(g) The proof of this case is in the text in subsection 3.2. QED.

Proof of Theorems 5 and 6:

The results can be obtained by using the following result in terms of expectations from Denuit, Eeckhoudt, Tsetlin and Walker (2010). Let $\tilde{x}, \tilde{y} \in [\underline{x}, \bar{x}]$ two real random variables. Then, $\tilde{x} \succ_{s-icv} \tilde{y}$ if and only if

$$E\left[\prod_{i=1}^{N} (c_i - \tilde{x}_i)_+^{k_i - 1}\right] \leq E\left[\prod_{i=1}^{N} (c_i - \tilde{y}_i)_+^{k_i - 1}\right],$$

for all $c_i \in [\underline{x}_i, \bar{x}_i]$ if $k_i = s_i$ and $c_i = \bar{x}_i$ if $k_i = 1, ..., s_i - 1; i = 1, ..., n$. Our calculus of these conditions using successive integrations by parts yield the results of the Theorems 5 and 6. Note that our stated result for Theorem 5 simplifies because $\triangle F(\bar{x}_1, \bar{x}_2) \leq 0$.

Proof of Theorem 7:

It is a classic result of one-dimensional stochastic dominance analysis (e.g., in Moyes, 1999b). Note that it can also be derived from Denuit-Eeckhoudt-Tsetlin-Walker's formula and integrations by parts as in the proof of Theorems 5 and 6.

Proof of Theorem 8:

(α) We have proven in Theorem 1 that one can use the intersection characterisation of generators for $s - idircv$ functions as the intersection of the sets of generators of some corresponding $(r_1, r_2) - icv$ functions. We apply it to $s = 4$. So far, this class of generator was unknown.

(β) Consider the representation of couple (x, y) in the complex plan $z \equiv x + iy \equiv \rho e^{i\theta}$, with the modulus of the complex number z defined as $\rho = \sqrt{x^2 + y^2}$ and its argument defined as $\theta = Arg(z)$, here restricted to $[0, \pi/2]$ so as to impose $x \geq 0$ and $y \geq 0$. The inverse transformation yields $x = \rho \cos\theta$ and $y = \rho \sin\theta$.

Then, the derivatives of a function $u(x, y)$ when transformed as a function of (ρ, θ) can be obtained by using the chain rule. For example, for a function let $u(\rho, \theta) \equiv u(x, y)$, thus allowing a slight abuse of notation to alleviate notations. By excluding the uninteresting case $\rho = 0$, where there is no two-side derivatives, we have $\frac{\partial u}{\partial \rho} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \rho} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \rho} = \frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta = \frac{1}{\rho}\left(x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right)$.

At the second order, by keeping the θ constant since we are calculating partial derivatives and θ does not depend on ρ in the formula, we obtain $\frac{\partial^2 u}{\partial \rho^2} = \frac{\partial\left[\frac{\partial u}{\partial x}\cos\theta + \frac{\partial u}{\partial y}\sin\theta\right]}{\partial \rho} = \cos\theta\frac{\partial\left[\frac{\partial u}{\partial x}\right]}{\partial \rho} + \sin\theta\frac{\partial\left[\frac{\partial u}{\partial y}\right]}{\partial \rho} = \cos^2\theta\frac{\partial^2 u}{\partial x^2} + 2\sin\theta\cos\theta\frac{\partial^2 u}{\partial x\partial y} + \sin^2\theta\frac{\partial^2 u}{\partial y^2}$ by replacing respectively u with $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$ in the previous calculus and rearranging. Iterating yields $\frac{\partial^3 u}{\partial \rho^3} = \cos^3\theta\frac{\partial^3 u}{\partial x^3} + 3\sin\theta\cos^2\theta\frac{\partial^3 u}{\partial x^2\partial y} + 3\sin^2\theta\cos\theta\frac{\partial^3 u}{\partial x\partial y^2} + \sin^3\theta\frac{\partial^3 u}{\partial y^3}$. Finally, $\frac{\partial^4 u}{\partial \rho^4} = \cos^4\theta\frac{\partial^4 u}{\partial x^4} + 4\sin\theta\cos^3\theta\frac{\partial^4 u}{\partial x^3\partial y} + 6\sin^2\theta\cos^2\theta\frac{\partial^4 u}{\partial x^2\partial y^2} + 4\sin^3\theta\cos\theta\frac{\partial^4 u}{\partial x\partial y^3} + \sin^4\theta\frac{\partial^4 u}{\partial y^4}$.

Then, it is clear in these formulae that a $(2,2) - icv$ utility function u in (x, y) has all its considered partial derivatives, with respect to x and y, alternatively non-positive and non-negative as we raise the order of derivation with respect to ρ (that is: positive for first order derivatives, negative for second order, etc). Since all the coefficients of these partials in these formulae are non-negative due to $\theta \in [0, \pi/2]$, we obtain that if function u is $(2,2) - icv$ in (x, y), then it is 4-idircv in ρ.

Let us now prove the reciprocal statement by recurrence, starting with the first-order derivatives. Let be a function $g(\rho, \theta)$ of (ρ, θ) and consider its variations after change in variables into (x, y). Assume that $\frac{\partial g}{\partial \rho} \geq 0$ for all $\rho > 0, \theta \in]0, \pi/2[$, so as to avoid boundaries where the derivatives of interest are not defined. Let us show that $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y} \geq 0$. Fixing $\theta = 0$ (respectively, $\theta = \pi/2$) yields $\rho = x$ (respectively $\rho = y$) and $\frac{\partial g}{\partial x} = \frac{\partial g}{\partial \rho}_{|\theta=0} \geq 0$ (respectively $\frac{\partial g}{\partial y} = \frac{\partial g}{\partial \rho}_{|\theta=\pi/2} \geq 0$), in this particular direction. Another way to see this result is just to notice that $\frac{\partial g}{\partial x}$ is the orthogonal projection of $\frac{\partial g}{\partial \rho}$ along the y-axis. The identity of the signs of $\frac{\partial g}{\partial y}$ and $\frac{\partial g}{\partial \rho}$ can be obtained in the same fashion.

Incrementing the derivation order with respect to ρ (i.e., imposing $\frac{\partial^2 g}{\partial \rho^2} \leq 0, \frac{\partial^3 g}{\partial \rho^3} \geq 0, \frac{\partial^4 g}{\partial \rho^4} \leq 0$) allows us to obtain the successive and respective non-positive and non-negative partials of order 2, 3 and 4 with respect to (x, y), as the consequence of iterating the previous reasoning by fixing $\theta = 0$ and $\theta = \pi/2$. We obtain $\frac{\partial^{k_1+k_2} g}{\partial x^{k_1}\partial y^{k_2}}$ of the sign of $(-1)^{k_1+k_2+1}$, as it is the sign of $\frac{\partial^{k_1+k_2} g}{\partial \rho^{k_1+k_2}}$. Therefore, $[u \text{ is } 4 - idircv \text{ in } (x, y)] \Longleftrightarrow [u \text{ is } 4 - icv \text{ in } \rho]$.

Finally, it is easy to obtain the stochastic dominance results of the proposition by applying already known results of one-dimensional stochastic dominance for $4 - icv$ utility functions and recalled in Theorem 7. QED.