The Role of Inequality in Poverty Measurement

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Introduction

Two forms of technologies for evaluating poverty

Unidimensional
- **Single** welfare variable – eg, calories
- Variables *can* be meaningfully **combined** – eg, expenditure

Multidimensional
- Variables **cannot** – eg, sanitation conditions and years of education
- Want variables **disaggregated** for policy – eg food and nonfood consumption
Introduction

**Demand** for *multidimensional* tools ↑

International organizations, countries

Literature has many measures


**Problems**

Inapplicable to *ordinal* variables

Found in multidimensional poverty

Or methods *extreme*

Union identification

Violates basic axioms
Introduction

New methodology Alkire-Foster (2011)
   Adjusted headcount ratio $M_0$ or MPI

Designed for ordinal variables
   Floor material

Has intermediate identification
   Dual cutoff approach

Satisfies key axioms
Introduction

Key axioms

**Ordinality**
- Can use with ordinal data

**Dimensional Monotonicity**
- Reflects deprivations of poor

**Subgroup Decomposability**
- Gauge contributions of population subgroups

**Dimensional Breakdown**
- Gauge contribution of dimensions

See example

Chad
Next week: UNDP and OPHI release newest global MPI results in NYC
Introduction

Critique

$M_0$ not sensitive to distribution among the poor

Axioms?

Some only for cardinal

Others weak: $\leq$ and not $<$.

$M_0$ satisfies!

Questions addressed here

Formulate strict axiom?

Construct measures satisfying this and other key properties?

Work in practice?
1. **Axioms**
   Ordinality, Dimensional Breakdown and Dimensional Transfer

2. **Class**
   
   M-Gamma $M_0$ for $\gamma \geq 0$
   
   $M_0^0 = H$ headcount ratio
   
   $M_0^1 = M_0$ adjusted headcount ratio
   
   $M_0^2$ squared count measure

3. **Impossibility**

4. **Resolution**
   
   Shapley Breakdown
   
   Use M-Gamma like P-alpha

5. **Application** Cameroon
Review: Poverty Measurement

Traditional two step framework of Sen (1976)

Identification Step “Who is poor?”
  Targeting

Aggregation Step “How much poverty?”
  Evaluation and monitoring
Unidimensional Poverty Measurement

**Identification step**

Typically uses **poverty line**

Poor if strictly below cutoff

Example: Distribution $x = (7,3,4,8)$ poverty line $\pi = 5$

Who is poor?

**Aggregation Step:**

Typically uses **poverty measure**

Formula aggregates data into poverty level
Unidimensional Poverty Measurement

FGT or P-alpha class

Incomes $x = (7, 1, 4, 8)$

Poverty line $\pi = 5$

Deprivation vector $g^0 = (0, 1, 1, 0)$

Headcount ratio $P_0(x; \pi) = H = \mu(g^0) = 2/4$

Normalized gap vector $g^1 = (0, 4/5, 1/5, 0)$

Poverty gap $P_1(x; \pi) = HII = \mu(g^1) = 5/20$

Squared gap vector $g^2 = (0, 16/25, 1/25, 0)$

FGT Measure $P_2(x; \pi) = \mu(g^2) = 17/100$

Note: All based on normalized gap $\frac{\pi - x_i}{\pi}$ raised to power $\alpha \geq 0$
Our Methodology

Alkire and Foster (2011)

Generalized FGT to multidimensional case

Dual cutoff identification

Deprivation cutoffs $z_1, \ldots, z_d$ within dimensions

Poverty cutoff $k$ across dimensions

Concept of poverty

A person is poor if multiply deprived enough

Consistent with

Cardinal and ordinal data

Union, Intersection, and indermediate identification

Example will clarify
Our Methodology

Achievement matrix with equally valued dimensions

\[
y = \begin{bmatrix}
13.1 & 14 & 4 & 1 \\
15.2 & 7 & 5 & 0 \\
12.5 & 10 & 1 & 0 \\
20 & 11 & 3 & 1 
\end{bmatrix}
\]

\[
z = (13, 12, 3, 1)
\]

Dimensions

Persons

Cutoffs
Our Methodology

Deprivation Matrix

Deprivation Score

\[
g^0 = \begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0
\end{bmatrix}
\]

\[c_i\]

0/4

2/4

4/4

1/4
Our Methodology

Deprivation Matrix

\[
g^0 = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 \\
1 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 \\
\end{bmatrix}
\]

Deprivation Score

\[
c_i
\]

<table>
<thead>
<tr>
<th>Deprivation Score</th>
<th>c_i</th>
</tr>
</thead>
<tbody>
<tr>
<td>0/4</td>
<td>0/4</td>
</tr>
<tr>
<td>2/4</td>
<td>2/4</td>
</tr>
<tr>
<td>4/4</td>
<td>4/4</td>
</tr>
<tr>
<td>1/4</td>
<td>1/4</td>
</tr>
</tbody>
</table>

**Identification:** Who is poor?

If poverty cutoff is \( k = 2/4 \), middle two persons are poor.
Our Methodology

Censored Deprivation Matrix

\[
\begin{bmatrix}
0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 1 & 0 \\
1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

Censored Deprivation Score

\[ c_i(k) \]

\[
\begin{align*}
g^0(k) &= \begin{bmatrix}
0/4 \\
2/4 \\
4/4 \\
0/4
\end{bmatrix}
\end{align*}
\]

Why censor? To focus on the poor, must ignore the deprivations of nonpoor.
Our Methodology

**Aggregation:** Adjusted Headcount Measure

\[ M_0 = \mu(g^0(k)) = \mu(c(k)) = \frac{3}{8} \]

\[ c_i(k) \]

\[ g^0(k) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \]

\[ M_0 = HA \] where

- H = multidimensional **headcount** ratio = \( \frac{1}{2} \) “incidence”
- A = average deprivation share among poor = \( \frac{3}{4} \) “intensity”

Note: Easily generalized to different weights summing to 1
Adjusted Headcount Ratio

Properties

Invariance Properties: Ordinality, Symmetry, Replication Invariance, Deprivation Focus, Poverty Focus

Dominance Properties: Weak Monotonicity, Dimensional Monotonicity, Weak Rearrangement, Weak Transfer

Subgroup Properties: Subgroup Consistency, Subgroup Decomposability, Dimensional Breakdown

Digression

Definitions of Ordinality and Dimensional Breakdown
Ordinality

Definition An *equivalent representation* rescales all variables and deprivation cutoffs.

**Ordinality** An equivalent representation leaves poverty unchanged.

Eg Change scale on self reported health from 1,2,3,4,5 to 2,3,5,7,9, and poverty level should be unchanged

Note

Measure violates if relies on scale or normalized gaps

$M_0$ satisfies
Dimensional Breakdown after identification has taken place and the poverty status of each person has been fixed, multidimensional poverty can be expressed as a weighted sum of dimensional components.

Note
Component function for $j$ depends only on dimension $j$ data

Breakdown formula for $M_0$

$$M_0 = \sum_j w_j H_j$$

or weighted average of censored headcount ratios

Example
<table>
<thead>
<tr>
<th>Indicator</th>
<th>Censored Headcount Ratio $H_j$</th>
<th>Dimensional Breakdown $w_jH_j$</th>
<th>Relative Contribution $w_jH_j/M_0$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Years of Schooling</td>
<td>16.7</td>
<td>2.8</td>
<td>11.2%</td>
</tr>
<tr>
<td>School Attendance</td>
<td>18.4</td>
<td>3.1</td>
<td>12.4%</td>
</tr>
<tr>
<td>Child Mortality</td>
<td>27.4</td>
<td>4.6</td>
<td>18.4%</td>
</tr>
<tr>
<td>Nutrition</td>
<td>18.3</td>
<td>3.1</td>
<td>12.3%</td>
</tr>
<tr>
<td>Electricity</td>
<td>37.3</td>
<td>2.1</td>
<td>8.4%</td>
</tr>
<tr>
<td>Sanitation</td>
<td>34.7</td>
<td>1.9</td>
<td>7.8%</td>
</tr>
<tr>
<td>Water</td>
<td>28.9</td>
<td>1.6</td>
<td>6.5%</td>
</tr>
<tr>
<td>Flooring</td>
<td>34.5</td>
<td>1.9</td>
<td>7.7%</td>
</tr>
<tr>
<td>Fuel</td>
<td>45.5</td>
<td>2.5</td>
<td>10.2%</td>
</tr>
<tr>
<td>Assets</td>
<td>23</td>
<td>1.3</td>
<td>5.2%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>24.8</td>
<td>100.0%</td>
</tr>
</tbody>
</table>
New Property

Recall property in Alkire-Foster (2011)

**Dimensional Monotonicity** Multidimensional poverty should rise whenever a poor person becomes deprived in an additional dimension.

New property

**Dimensional Transfer** Multidimensional poverty should fall as a result of a dimensional rearrangement among the poor.

**A dimensional rearrangement among the poor** An association-decreasing rearrangement among the poor (in achievements) that is simultaneously an association-decreasing rearrangement in deprivations.
New Property

Example with \( z = (13,12,3,1) \)

<table>
<thead>
<tr>
<th>Achievements</th>
<th>Deprivations</th>
</tr>
</thead>
</table>
| \[
\begin{bmatrix}
12 & 13 & 2 & 1 \\
10 & 7 & 1 & 0 
\end{bmatrix}
\rightarrow
\begin{bmatrix}
12 & 7 & 2 & 1 \\
10 & 13 & 1 & 0 
\end{bmatrix}
\] | \[
\begin{bmatrix}
1 & 0 & 1 & 0 \\
1 & 1 & 1 & 0 
\end{bmatrix}
\rightarrow
\begin{bmatrix}
1 & 1 & 1 & 0 \\
1 & 0 & 1 & 1 
\end{bmatrix}
\] |

Dominance   No dominance   Dominance   No dominance

Dimensional Transfer implies poverty must **fall**

Note: Adjusted Headcount \( M_0 \)

Just *violates* Dimensional Transfer

Same average deprivation score

Question: Are there measures satisfying DT?
**M-Gamma Class** $M^\gamma_0$

**Identification:** Dual cutoff

**Aggregation:**

$$M^\gamma_0 = \mu(c^\gamma(k)) \quad \text{for } \gamma \geq 0$$

where $c^\gamma_i(k)$ is the censored deprivation score for person $i$ raised to the $\gamma$ power

**Note:** Based on “normalized attainment gap”

$$c^\gamma_i(k) = \left(\frac{d-a_i}{d}\right)^\gamma \quad \text{for poor } i$$

$$c^\gamma_i(k) = 0 \quad \text{for nonpoor } i$$

where $a_i$ is person $i$’s attainment score
M-Gamma Class $M_0^{\gamma}$

Main measures

$\gamma = 0$ headcount ratio $M_0^0 = H$
$\gamma = 1$ adjusted headcount ratio $M_0^1 = M_0$
$\gamma = 2$ squared count measure $M_0^2$

Note: Multidimensional analog to P-alpha

Dimensional Transfer satisfied for $\gamma > 1$ ✔
But Dimensional Breakdown violated for $\gamma > 1$ ✗
Recall

Dimensional Breakdown: $M$ can be expressed as a weighted average of component functions (after identification)

**Why does $M_0^2$ violate?**

Marginal impact of each dimension depends on all dimensions

**Question:** Any other measures satisfy both?

**Proposition** There is no symmetric multidimensional measure satisfying both Dimensional Breakdown and Dimensional Transfer

**Proof**

Follows Pattanaik et al (2012)

Idea: DT requires fall in poverty; DB requires unchanged
Importance of Dimensional Breakdown

**Coordination** of Ministries

Coordinated dashboard of censored headcount ratios

**Governance**

Stay the course in bad financial times

**Policy Analysis**

Composition of poverty across groups, space, and time

**Conclusion**

**Easy to construct** measure satisfying Dimensional Transfer

But at a **cost**: lose Dimensional Breakdown
1. Use multiple measures?
   M-gamma class analogous to P-alpha class ✔

2. Relax Dimensional Transfer?
   Already weak ✗

3. Relax Dimensional Breakdown?
   Already weak ✗
   Datt (2017) suggests Shapley methods
Shapley Breakdown

Shapley Value

Finds contributions of parts to whole

Especially useful for nonlinear functions: \( M_0^\gamma \) for \( \gamma \neq 1 \)

Example: One person, 10 indicators and union ident.

Poverty is censored deprivation score to \( \gamma \) power: \( (c_i(k))^\gamma \)

If not poor, then total and parts are zero
If poor and not deprived in \( j \), then \( j \) has zero contribution
If poor and deprived in \( j \), then the marginal impact of \( j \) depends on which dimensional indicator goes first, second, third…

Shapley: average marginal product across all permutations

Tedious to calculate

No intuitive link for policy

Makes no sense for hierarchical indicators
Shapley Breakdown

Example: Pat

Ten Indicators
- Nutrition
- Child Mortality
- Years of Schooling
- Children enroled
- Cooking Fuel
- Sanitation
- Water
- Electricity
- Floor
- Assets
Shapley Breakdown

Example: Jo

Three Dimensions of Poverty

Ten Indicators
- Nutrition
- Child Mortality
- Years of Schooling
- Children enrolled
- Cooking Fuel
- Sanitation
- Water
- Electricity
- Floor
- Assets
Inconsistency due to hierarchical variables

Relative Contribution of Indicator 1 as Gamma Varies

But ok for $\gamma = 2$
Results

Definition

Consider set of people **poor** and **deprived** in j

Censored intensity $A_j = \text{average intensity or breadth of poverty in this group}$

Recall

Censored headcount ratio $H_j = \text{incidence of this group in overall population}$

**Theorem** The Shapley breakdown for $M_0^2$ has a closed form solution. Each component is obtained by multiplying each component of the dimensional breakdown of $M_0$ by $A_j$. 
Dimensional and Shapley Breakdown – Cameroon

Table V: Breakdowns of $M_0^1$ and $M_0^2$

<table>
<thead>
<tr>
<th>Indicator</th>
<th>$H_j$</th>
<th>$w_jH_j$</th>
<th>$A_j$</th>
<th>$M_{0j}$</th>
<th>$w_jM_{0j}$</th>
<th>$w_jH_j/M_0^1$</th>
<th>$w_jM_{0j}/M_0^2$</th>
<th>Percentag e Point Diff</th>
</tr>
</thead>
<tbody>
<tr>
<td>Schooling</td>
<td>16.7</td>
<td>2.8</td>
<td>66.2</td>
<td>11.1</td>
<td>1.8</td>
<td>11.2%</td>
<td>12.5%</td>
<td>1.3</td>
</tr>
<tr>
<td>Attendance</td>
<td>18.4</td>
<td>3.1</td>
<td>66.1</td>
<td>12.2</td>
<td>2.0</td>
<td>12.4%</td>
<td>13.8%</td>
<td>1.4</td>
</tr>
<tr>
<td>Child Mortality</td>
<td>27.4</td>
<td>4.6</td>
<td>57.4</td>
<td>15.7</td>
<td>2.6</td>
<td>18.4%</td>
<td>17.8%</td>
<td>-0.6</td>
</tr>
<tr>
<td>Nutrition</td>
<td>18.3</td>
<td>3.1</td>
<td>63.9</td>
<td>11.7</td>
<td>1.9</td>
<td>12.3%</td>
<td>13.3%</td>
<td>1.0</td>
</tr>
<tr>
<td>Electricity</td>
<td>37.3</td>
<td>2.1</td>
<td>56.4</td>
<td>21.0</td>
<td>1.2</td>
<td>8.4%</td>
<td>7.9%</td>
<td>1.0</td>
</tr>
<tr>
<td>Sanitation</td>
<td>34.7</td>
<td>1.9</td>
<td>53.9</td>
<td>18.7</td>
<td>1.0</td>
<td>7.8%</td>
<td>7.1%</td>
<td>-0.7</td>
</tr>
<tr>
<td>Water</td>
<td>28.9</td>
<td>1.6</td>
<td>56.1</td>
<td>16.2</td>
<td>0.9</td>
<td>6.5%</td>
<td>6.1%</td>
<td>-0.4</td>
</tr>
<tr>
<td>Flooring</td>
<td>34.5</td>
<td>1.9</td>
<td>56.4</td>
<td>19.5</td>
<td>1.1</td>
<td>7.7%</td>
<td>7.4%</td>
<td>-0.4</td>
</tr>
<tr>
<td>Fuel</td>
<td>45.5</td>
<td>2.5</td>
<td>54</td>
<td>24.6</td>
<td>1.4</td>
<td>10.2%</td>
<td>9.3%</td>
<td>-0.9</td>
</tr>
<tr>
<td>Assets</td>
<td>23</td>
<td>1.3</td>
<td>57</td>
<td>13.1</td>
<td>0.7</td>
<td>5.2%</td>
<td>5.0%</td>
<td>-0.2</td>
</tr>
</tbody>
</table>

$M_0^1 = 24.8$  
$M_0^2 = 14.7$

Note similarity of relative contributions
Conclusion

Derived closed form solution for Shapley breakdown of squared count measure

Should use the three main M-gamma measures in tandem analogous to P-alpha measures

$M_0$ as the central measures for analysis satisfying Dimensional Breakdown and just violating Dimensional Transfer

$H$ as a key partial measure of incidence of poverty

$M_0^2$ (and its Shapley breakdown) to evaluate the effects of inequality among the poor
Thank you!