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Shadow prices for a small open economy under uncertainty

Which expected values are valid

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Abstract: This paper re-examines the validity of using expected values to evaluate the social profitability of public investments under uncertainty. Departing from the usual assumption of an aggregate good, the setting is a small open economy that faces stochastic world prices for tradable goods and productivity levels in domestic production. It is shown that the so-called 'border price rule'—the vector of the shadow prices of traded goods is a scalar multiple of their world price vector—is, in general, invalid when the vector of mean world prices is used. The rule in that form is valid when the coefficient of risk aversion is 1. The error involved is small, however, even for values of that coefficient far from 1, including risk neutrality, when public expenditures are financed by lump-sum taxes. It is also small when preferences over goods are Cobb-Douglas and revenues are raised by taxing commodities. In contrast, using expected values to derive the shadow wage rate results in quite substantial errors when risk aversion is strong.

Keywords: uncertainty, expected values, shadow prices, public investment
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1 Introduction

When is it valid to use expected values in order to assess whether a project is socially profitable, in the sense of improving social welfare? In a well-known paper, Arrow and Lind (1970) prove that if a project's net returns are independent of national income and distributed in lump-sum form over a sufficiently large population, then the aggregate cost of the risk associated with the project is negligible, so that the project should be accepted if its expected net return is positive and rejected otherwise. Little and Mirrlees (1974: 328-331) arrive at the same result by a somewhat different route, but without imposing the condition that the project be very small, while elucidating why the condition that the project's net returns be independent of national income is vital. They also make a prefatory remark in connection with the assumption that there is a single commodity: 'It is not so clear that this assumption is innocuous [...]' (p. 329). This provides the point of departure for the present paper.

The vast literature on shadow prices for project evaluation in developing countries is overwhelmingly concerned with settings in which there are many goods, drawing a careful distinction between those that are internationally tradable and those that are not. The shadow wage rate and the social discount rate naturally have a prominent place, too. Almost invariably absent, however, is any form of uncertainty. Squire (1989), in a thorough and insightful survey, lists a very few exceptions of the case-study kind, but otherwise refers the reader to such general contributions as Arrow and Lind (1970) and Graham (1981). In their authoritative survey of the theory of cost-benefit analysis, Drèze and Stern (1987) also pay rather little attention to uncertainty. They do, however, construct an illuminating and simple example in which peasants supply a fixed amount of labour to cultivation and public projects combined, with both activities yielding risky output of an aggregate good. Since the marginal product of labour in cultivation will be low when the marginal utility of income is high, as for example in a drought, the shadow price of labour will be lower than the expected marginal product of labour in cultivation, thus illustrating the potential importance of taking covariance terms into account.

The chief object of this paper, therefore, is to derive shadow prices for goods and labour in a small open economy that is beset by stochastic world prices for traded goods and stochastic levels of productivity in domestic production. These exogenous sources of uncertainty result in stochastic domestic prices for all goods and labour. A question of central importance is whether the so-called 'border price' rule holds for traded goods: that is to say, is the vector of the shadow prices of such goods equal to a scalar multiple of the vector of their world prices? It will be shown that while this classic rule indeed holds for *contingent* shadow prices, it is invalid, in general, for the purposes of appraising projects *ex ante*, when the state in which the delivery or use of a good occurs has yet to be revealed. This finding implies, in particular, that it is generally invalid to use the expected values of traded goods' world prices in order to estimate their relative shadow prices for certain delivery, even if households are risk neutral. In consequence, the use of mean values to estimate the corresponding shadow prices of non-tradables and labour is likewise generally invalid. For the family of preferences over goods whose indirect utility function has as its sole argument the level of income deflated by an exact price index, the only exception arises when the coefficient of relative risk aversion is unity: only in this borderline case is it strictly valid to use the mean values of world prices. A fairly general argument, supported by some numerical examples, serves to establish, however, that the rule is robust. Given the ranges of values that exogenously stochastic variables take in practice, even substantial departures of the coefficient of relative risk aversion from unity will result in small errors when government expenditures are financed by lump-sum taxes. The same holds when commodity taxes are employed and preferences over goods are Cobb-Douglas, or almost so. The corresponding error in the shadow wage rate, in contrast, is rather large when risk aversion is strong.

Turning to individual projects, not only are these undertaken in an uncertain general environment, but they are also subject to idiosyncratic risks, in the sense that they may not function exactly as planned, even for each given realisation of the set of variates describing the economy's stochastic setting. This raises the question of whether it is valid to appraise a project's profitability on the basis of the expected values of its inputs and outputs. If the idiosyncratic risks are independent of the systemic ones, intuition suggests that such a procedure is indeed valid, but this conjecture still needs formal investigation. If, on the contrary, the two kinds of risks are not independent, then neglecting the covariance terms may lead to serious errors. This consideration may be important in practice. To give an example, not only will an agricultural project's performance almost surely be influenced by the growing conditions that affect the whole sector, but it may also be more susceptible to adverse conditions.

The plan of the paper is as follows. The model is set out in Section 2, followed by a proof of the main result concerning the validity of the border price rule. Section 3 analyses shadow prices when the coefficient of relative risk aversion is constant and lump-sum taxes are available. There follow some numerical examples illustrating the magnitude of the error that results from using mean values when there are CES-preferences over goods. The analysis is extended to commodity taxes in Section 4. The topic of risks specific to a project, as opposed to those in the environment, is taken up in Section 5, also with numerical examples. The paper concludes with a brief discussion.

2 The Model

The model is essentially that in Bell and Devarajan (1983), extended here to a setting in which the world prices of tradable goods and the levels of productivity in domestic production activities are stochastic. The technologies in the private sector exhibit constant returns to scale, with labour as the sole input. Households supply their endowments of labour completely inelastically. There are three private goods: 1 and 2 are tradable at parametric world prices; good 3 is non-tradable. All markets are perfect. Since labour is the only primary factor and there are constant returns to scale, only one of the tradables will be produced domestically. In the stochastic setting examined here, which of the two is imported will depend on the realised state. A precise condition will be derived below.

The government raises revenue in order to finance the provision of a public good in some fixed amount. It must augment any profits from public production by taxing wages or goods, there being no private profits to tax. It should be noted that since the entire factor endowment is supplied completely inelastically, a tax on wages or a proportional tax on all goods at the same rate would be effectively lump-sum in nature.

Let the variate P_i^* $(i = 1, 2)$ denote the world price of good i and L_i $(i = 1, 2, 3)$ the corresponding unit labour requirement in production. These variates and their associated distributions constitute the stochastic elements in the set of exogenous variables. Denote by $S(P_i^*)$ the set of all possible values of P_i^*; $S(L_i)$ is analogously defined. The set of all possible realisations is then

$$S = S(P_1^*) \times S(P_2^*) \times S(L_1) \times S(L_2) \times S(L_3).$$

Together with the behaviour of the economy's agents, the said variates induce the wage rate and the price of the non-tradable, which are likewise stochastic, but endogenous; they are denoted by the variates W and P_3, respectively. The following assumption is made for simplicity:

Assumption 1. Good 1 will be produced domestically and good 2 wholly imported for all realisations p_i^* and $l_i,\ i = 1, 2$.

In the absence of taxes on tradable goods, their domestic prices will be equal to their respective world prices:

$$p_i = q_i = p_i^*, \quad i = 1, 2, \tag{1}$$

where p_i and q_i denote the producer and consumer price of good i, respectively.

The unit cost of producing good i domestically is the variate WL_i.

Assumption 2. Inputs of labour must be chosen before the state $s \in S$ is revealed. Wages are paid out of realised revenues following production.

The realised wage rate, w, therefore adjusts to the realised level l_i ($i = 1, 2$), and the resulting price of output is such that profits in equilibrium are zero *ex post.* Hence, good i will be produced domestically (and exported), and $j \neq i$ will be wholly imported, if $wl_i \leq p_i^*$ and $wl_j > p_j^*$. If i is produced domestically, then

$$w = p_i^*/l_i \ \text{ and } \ p_i^*/l_i > p_j^*/l_j. \tag{2}$$

In virtue of Assumption 1, we have

$$w = p_1^*/l_1 \ \text{ and } \ p_1^*/l_1 > p_2^*/l_2 \quad \text{for all realisations of } P_i^* \text{ and } L_i \ (i = 1, 2).$$

The producer price of the non-tradable is likewise equal to the unit cost of producing it. For the realisations p_1^*, l_1 and l_3, we have

$$p_3 = wl_3 = p_1^* l_3/l_i. \tag{3}$$

To sum up, the vector of producer prices under the above assumptions is

$$(\mathbf{p}, w) = p_1^* \cdot (1, p_2^*/p_1^*, l_3/l_1, 1/l_1), \tag{4}$$

where $(p_1^*, p_2^*, l_1, l_2, l_3)$ is the realisation from the set S. That producer prices are independent of quantities is a particularly convenient feature of the model. In what follows, it will useful emphasise that all variables are, in principle, dependent on the particular draw from the set S by introducing $s \in S$ explicitly into the notation. The absence of s in parentheses, (s), indicates that the variable in question is non-stochastic.

The next step is to establish the conditions for markets to clear in equilibrium. Let $x_i(s)$ and $y_i(s)$, respectively, denote the private consumption and production of good i in state s; analogously, $\mathbf{z}(s) = (z_1(s), z_2(s), z_3(s), z_l(s))$ denotes the public sector's net supply of goods and labour; and $e_i(s)$ denotes the net exports of good i. The market-clearing equations in state s are

$$y_1(s) + z_1(s) = x_1(s) + e_1(s), \tag{5}$$

$$z_2(s) = x_2(s) + e_2(s), \tag{6}$$

$$y_3(s) + z_3(s) = x_3(s), \tag{7}$$

and

$$\omega_l + z_l(s) - g_l = l_1(s)y_1(s) + \ l_3(s)y_3(s), \tag{8}$$

where ω_l is the economy's aggregate endowment of labour, g_l is the input thereof needed to produce the public good, and $z_l(s)$ is the government's supply thereof net of g_l. It then follows from the fact that both the private and public sectors are on their respective budget lines that the economy's trade deficit at world prices is equal to its endowment of foreign exchange, ω_f:[1]

$$p_1^*(s)\,e_1(s) + p_2^*(s)\,e_2(s) + \omega_f = 0. \tag{9}$$

This statement of Walras's law is a convenient way of deriving the shadow price of public income.

Households' aggregate gross income in state s is $w(s)\,\omega_l = (p_1^*(s)/l_1(s))\,\omega_l$. Let the associated indirect utility function be denoted by $v(\mathbf{q}(s), m(s))$, where $m(s) = w(s)\,\omega_l - t(s)$ is the corresponding level of income after tax. Faced with the set of outcomes arising from S, let households' preference functional over the lotteries in question be represented by $V = E_s[v(\mathbf{q}(s), m(s))] + \psi(g_l)$, where the term $\psi(g_l)$ is a constant, given any particular choice of g_l, and E is the expectation operator.

The government can choose the tax vector (t_1, t_2, t_3, t) only after s has been revealed. Its decision problem is then

$$\max_{(t_1,t_2,t_3,t|s)} \; v(s) \quad \text{s.t. } (5) - (9). \tag{10}$$

Writing the Lagrangian in the following form (Drèze and Stern, 1987), we then employ

[1] See Bell (2003: 244-5) for the details of the derivation.

the envelope theorem to obtain all shadow prices, *contingent* on s:

$$\begin{aligned}\Phi(s) &= v(s) + \sum_i \lambda_i(s)[y_i(s) + z_i(s) - x_i(\mathbf{q}(s), m(s)) - e_i(s)] \\ &+ \lambda_l(s)[\omega_l + z_l(s) - l_1(s)y_1(s) - l_3(s)y_3(s) - g_l] \\ &+ \mu(s)[\omega_f + p_1^*(s)\, e_1(s) + p_2^*(s)\, e_2(s)]. \end{aligned} \tag{11}$$

The *contingent* shadow prices are the respective changes in $v(s)$ resulting from marginal changes in the government's net supply vector in state s:

$$\pi_i(s) \equiv \frac{\partial \Phi^0(s)}{\partial z_i(s)} = \frac{\partial v^0(s)}{\partial z_i(s)} = \lambda_i(s), \quad i = 1, 2, 3, l \tag{12}$$

$$\pi_f(s) \equiv \frac{\partial \Phi^0(s)}{\partial \omega_f} = \frac{\partial v^0(s)}{\partial \omega_f} = \mu(s), \tag{13}$$

where the superscript '0' refers to the optimum of problem (10) and it should be noted that the contingent shadow wage rate is $\lambda_l(s)$, which is the reduction in $v^0(s)$ when labour is employed in, not produced by, the public sector. The contingent shadow price of public income is $\mu(s)$.

The appraisal of public projects, however, must be made *ex ante*, before the state s is revealed. For this purpose, the shadow prices of goods, labour and public income relate to changes in welfare as given by the functional V. That is to say, they are the effects on welfare of marginal deliveries made whatever be the realisation $s \in S$: $dz_i(s) = dz_i \, \forall s \in S, i = 1, 2, 3, l$.[2] The said shadow prices are therefore the expected values of their respective contingent shadow prices. We have

$$\pi_i \equiv \frac{\partial V^0}{\partial z_i} = E_s\left[\frac{\partial v^0(s)}{\partial z_i(s)}\right] = E_s[\lambda_i(s)], \quad i = 1, 2, 3, l \tag{14}$$

[2]This does not rule out the possibility of idiosyncratic risks specific to individual projects. See Section 5.

and

$$\pi_f \equiv \frac{\partial V^0}{\partial \omega_f} = E_s\left[\frac{\partial v^0(s)}{\partial \omega_f}\right] = E_s[\mu(s)]. \tag{15}$$

At the optimum, $\Phi(s)$ must be stationary w.r.t. all endogenous variables, whether they be chosen by the government or adjust to bring about equilibrium. The derivatives w.r.t. $\mathbf{e}(s)$ and $\mathbf{y}(s)$ yield

$$\frac{\partial \Phi^0(s)}{\partial e_i(s)} = -\lambda_i(s) + \mu(s)\, p_i^*(s), \quad i = 1, 2, \tag{16}$$

and

$$\frac{\partial \Phi^0(s)}{\partial y_i(s)} = \lambda_i(s) - l_i(s)\lambda_l(s), \quad i = 1, 3. \tag{17}$$

Eq. (16) yields the celebrated border price rule, albeit expressed as contingent on the state s: the contingent shadow prices of traded goods are proportional to their respective world prices, the factor of proportionality being the contingent shadow price of public income. Eq. (17) states that the contingent shadow price of a good produced domestically is equal to the contingent marginal shadow cost of producing it. Hence, the vector of contingent shadow prices is proportional to the vector of contingent producer prices:

$$\boldsymbol{\pi}(s) = \mu(s) \cdot p_1^*(s)\left(1, \frac{p_2^*(s)}{p_1^*(s)}, \frac{l_3(s)}{l_1(s)}, \frac{1}{l_1(s)}\right). \tag{18}$$

From (14) and (15), the corresponding shadow prices for project appraisal *ex ante* are

$$\pi_i = E_s[\mu(s)\, p_i^*(s)] = E_s[\mu(s)] \cdot E_s[p_i^*(s)] + \mathrm{cov}[\mu(s), p_i^*(s)],\ i = 1, 2, \tag{19}$$

$$\begin{aligned} \pi_3 &= E_s[\mu(s) \cdot p_1^*(s)\, l_3(s)/l_1(s)] \\ &= E_s[\mu(s)] \cdot E_s[p_1^*(s)\, l_3(s)/l_1(s)] + \mathrm{cov}[\mu(s), p_1^*(s)\, l_3(s)/l_1(s)], \end{aligned} \tag{20}$$

and

$$\pi_l = E_s[\mu(s) \cdot p_1^*(s)/l_1(s)] = E_s[\mu(s)] \cdot E_s[p_1^*(s)/l_1(s)] + \text{cov}[\mu(s), p_1^*(s)/l_1(s)]. \quad (21)$$

It is seen that the magnitude of the covariance of the shadow price of public income and the exogenous stochastic variables determines the size of the error, if any, in using the mean values of the latter to estimate shadow prices. Suppose, therefore, that the world price of good 1 does not vary ($p_1^*(s) = p_1^* \ \forall s$), so that $E_s[p_1^*(s)] = p_1^*$, $\text{cov}[\mu(s), p_1^*(s)] = 0$ and $\pi_1 = E_s[\mu(s)] \cdot p_1^*$. Then substituting into (19), we obtain

$$\pi_2 = \pi_1 \cdot \frac{E_s[p_2^*(s)]}{p_1^*} + \text{cov}[\mu(s), p_2^*(s)],$$

where the covariance term is, in general, non-zero if the world price of good 2 is stochastic. The same applies, *mutatis mutandis*, if the roles of the goods are reversed. This establishes the first result.

Proposition 1. If the world price of either traded good is stochastic and these prices are not perfectly correlated, then the ratio of their shadow prices for ex ante appraisal is not, in general, equal to the ratio of their mean world prices.

That is to say, the so-called border price rule is not, in general, correct when the mean world price is employed. The same holds for the use of mean values to derive the shadow prices of the non-tradable and labour for *ex ante* appraisal purposes.

One always has the choice of numéraire. The endowment ω_f corresponds to foreign exchange in the hands of the government, which is the choice in Little and Mirrlees (1974). In what follows, it will simplify matters to set this endowment to zero and choose a good instead. A natural choice is a tradable good. Let it be good 1, so that $p_1^*(s) = 1 \ \forall s$, which yields the (stochastic) barter terms of trade, $p_1^*(s)/p_2^*(s) = 1/p_2^*(s)$, as a normalisation. The Lagrange multiplier $\mu(s)$ continues to play a vital role, however, whereby its precise value arises from the normalisation $\omega_f = 0$.

The above assumptions about households' preferences, as represented by V, are quite weak. It will be proved in the next section that even with lump-sum taxation, only one member of a whole family of preferences yields an exception to the generality of Proposition 1: in this particular case, the border-price rule indeed holds using the expected values of world prices. For all other members of the family, an error will result, whose size will be analysed in detail.

3 Expected Values and Shadow Prices

The indirect utility function v is quasi-convex in $(\mathbf{q}(s),\, m(s))$ and homogeneous of degree zero in $\mathbf{q}(s)$ and $m(s)$. Let preferences satisfy the following assumption:

Assumption 3. $v(\mathbf{q}(s),\, m(s)) = \phi\,[m(s)/\kappa(\mathbf{q}(s))]$, where ϕ is increasing in the argument $m(s)/\kappa(\mathbf{q}(s))$ and differentiable, and κ is increasing, differentiable and homogeneous of degree 1 in $\mathbf{q}(s)$.

Remark 1. The function κ can be thought of as yielding the true price level corresponding to the preferences underlying v. The quantity $m(s)/\kappa(\mathbf{q}(s))$ is the corresponding level of real income at prices $\mathbf{q}(s)$. Put slightly differently, given some reference level of utility u and the price vectors $\mathbf{q}$ and $\mathbf{q}'$, let $c(\mathbf{q}, u)$ and $c(\mathbf{q}',\, u)$, respectively, denote the minimum cost of attaining u at the said prices. Then, taking $\mathbf{q}$ as the reference price vector, the Könus price index is defined to be $c(\mathbf{q}',\, u)/c(\mathbf{q}, u)$ (Diewert, 1988).

Inspection of (19) – (21) reveals that a potentially promising approach to establishing the signs of the covariance terms is to examine whether the functions involved are concave or convex for all positive prices; for Jensen's inequality will then yield the sign in question. To this end, the following lemma will be helpful.

Lemma 1. If $g(\mathbf{x})$ is concave in $\mathbf{x}$, then $1/g(\mathbf{x})$ is strictly convex.

Proof: see Appendix.

Since the true cost-of-living index κ is homogeneous of degree 1 in $\mathbf{q}(s)$, ϕ is strictly convex in $\mathbf{q}(s)$. This fact turns out to exert an important influence on shadow prices when consumer prices are stochastic.

It is well known that shadow prices depend on the government's choice of policies to bring about (9). In the present setting, its choice is the vector of taxes. We begin with the simplest case, i.e., lump-sum taxes, deferring commodity taxes to Section 4.

3.1 Lump-sum taxes and constant relative risk aversion

Let wages, but not goods, be taxed, so that $\mathbf{q}(s) = \mathbf{p}(s)$ and $m(s) = w(s)\,\omega_l - t(s)$. Suppose the government's endowment of foreign exchange ω_f were to increase. It would then distribute this windfall by decreasing $t(s)$ in the same amount; and under the above assumptions, this action would leave producer prices, and hence consumer prices, unchanged. It follows from (13) that $\partial v^0(s)/\partial m(s) = \mu(s)$. Recalling Assumption 3 and substituting into (19), we obtain

$$\pi_i = E_s \left[\frac{\partial \phi[m(s)/\kappa(\mathbf{q}(s))]}{\partial m(s)} \cdot p_i^*(s) \right], \; i = 1, 2. \tag{22}$$

If households are risk neutral, ϕ is affine in $m(s)/\kappa(\mathbf{q}(s))$ and (22) specialises to

$$\pi_1 = E_s \left[\frac{p_1^*(s)}{\kappa(\mathbf{q}(s))} \right] = E_s \left[\frac{1}{\kappa(1, p_2^*(s)/p_1^*(s), l_3(s)/l_1(s))} \right],$$

and

$$\pi_2 = E_s \left[\frac{p_2^*(s)}{\kappa(\mathbf{q}(s))} \right] = E_s \left[\frac{1}{\kappa(p_1^*(s)/p_2^*(s), 1, p_1^*(s)l_3(s)/p_2^*(s)l_1(s))} \right].$$

The level of income $m(s)$ is not in play. Choosing good 1 as numéraire, i.e., $p_1^*(s) = 1 \, \forall s$, we have

$$\pi_2 = E_s \left[\frac{p_2^*(s)}{\kappa(\mathbf{q}(s))} \right] = E_s[p_2^*(s)] \cdot \pi_1 + \mathrm{cov}\left(p_2^*(s), 1/\kappa(\mathbf{q}(s))\right) < E_s[p_2^*(s)] \cdot \pi_1, \qquad (23)$$

where the inequality follows from the fact that $\kappa(\mathbf{q}(s))$ is increasing in $p_2^*(s)$. Hence, the so-called border price rule is incorrect when the mean world price is employed. Expressed more precisely, we have established

Proposition 2. If households are risk neutral and good 1 is the numéraire, then under Assumption 3, $\pi_2/\pi_1 < E_s[p_2^*/p_1^*]$.

Fluctuations in prices matter, therefore, even under risk-neutrality. Since κ is strictly concave in any two of its arguments, it follows from Lemma 1 and Jensen's inequality that $\pi_i = E_s[\mu(s)\, p_i^*(s)] < E_s[\mu(s)] \cdot E_s[p_i^*(s)]$ if $p_i^*(s)$ depends on s. With good 1 as numéraire, $\pi_1 = E_s[\mu(s)]$; but $p_2^*(s)/\kappa(\mathbf{q}(s))$ is strictly convex in $p_2^*(s)$.

An analogy provides some intuition for this finding. A well-known property of the competitive firm's profit function is that it is convex in prices. This implies that expected profits with variable prices exceed the level attained when prices take their mean values if the said function is strictly convex. Likewise, risk-neutral consumers in the present setting also prefer variable prices to the mean, all else being equal. In a world with but one consumer good, however, this effect is ruled out; and with lump-sum taxes, changes in the public sector's net supply vector result in changes in private income m that are independent of s.

The size of the error, expressed proportionally, involved in using the mean values of world prices is $\mathrm{cov}(p_2^*(s), 1/\kappa(\mathbf{q}(s))/E_s[1/\kappa(\mathbf{q}(s))]$. This is likely to be rather small. For the prices of goods 1 and 3 also enter into $\kappa(s)$, and if the taste for good 2 is not very strong, substitution possibilities will also work to restrict the deviation of each $1/\kappa(\mathbf{q}(s))$ from the mean, $E_s[1/\kappa(\mathbf{q}(s))]$. This matter will be pursued further in Section 3.2.

Turning to the shadow price of labour, we have

$$\begin{aligned}\pi_l &= E_s\left[\frac{1/l_1(s)}{[\kappa(1, p_2^*(s)/p_1^*(s), l_3(s)/l_1(s))]}\right] \\ &= \pi_1 \cdot E_s[1/l_1(s)] + \text{cov}\left[\frac{1}{l_1(s)}, \frac{1}{\kappa(1, p_2^*(s)/p_1^*(s), l_3(s)/l_1(s))}\right]. \qquad (24)\end{aligned}$$

$E_s[1/l_1(s)]$ is the mean level of efficiency in the export sector, and although π_1 fully reflects all the influences considered above, it is seen that the shadow wage rate is equal to their product plus the covariance of the *contingent* shadow price of good 1 and the level of productivity, $1/l_1(s)$. It is quite plausible that the latter is independent of world prices when producer prices are independent of quantities – recall (4); but κ also depends on the level of domestic productivity $1/l_1(s)$, so that the covariance term is almost surely not equal to zero. Once again, therefore, the use of mean values, even with the correct shadow prices of tradable goods, is very likely to result in an error.

With risk-neutrality as a benchmark, we turn to risk-averse households. By continuity, the above results also hold whenever households are only mildly risk-averse. Risk-aversion, however, implies that ϕ is strictly concave, thus offsetting the advantages of variability stemming from the strict convexity of $1/\kappa(\mathbf{q}(s)$. Are there constellations in which the two effects exactly cancel out? In order to address this question, let the coefficient of relative risk aversion be constant.

Assumption 4. Let

$$\phi = (1-\rho)^{-1}\left(\frac{m(s)}{\kappa(\mathbf{q}(s))}\right)^{1-\rho}, \ \rho \geq 0,$$

where ρ is to be interpreted as the coefficient of relative risk aversion.

Given this additional restriction on preferences, we have

$$\pi_i = E_s\left[\frac{[m(s)]^{-\rho}}{\kappa[(\mathbf{q}(s))]^{1-\rho}}\cdot p_i^*(s)\right],\ i=1,2.$$

Since consumer prices are equal to producer prices and κ is homogeneous of degree 1, it follows from (4) that this may be expressed as

$$\pi_i = E_s\left[\frac{[m(s)]^{-\rho}\cdot p_i^*(s)}{[p_1^*(s)\kappa(1, p_2^*(s)/p_1^*(s), l_3(s)/l_1(s))]^{1-\rho}}\right],\ i=1,2. \tag{25}$$

From (21), the shadow value of labour is

$$\pi_l = E_s\left[\frac{[m(s)]^{-\rho}\cdot p_1^*(s)/l_1(s)}{[p_1^*(s)\kappa(1, p_2^*(s)/p_1^*(s), l_3(s)/l_1(s))]^{1-\rho}}\right]. \tag{26}$$

By inspection, the special case $\rho = 1$, i.e., $\phi = \ln[m(s)/\kappa(\mathbf{q}(s))]$, attracts attention. Eq. (25) then yields the strikingly simple expression

$$\pi_i = E_s\left[\frac{p_i^*(s)}{m(s)}\right],\ i=1,2,$$

from which κ is absent. The level of income is $m(s) = (p_1^*(s)/l_1(s))\,\omega_l - t(s)$. By assumption, the production of the public good requires g_l units of labour in all states s. Thus, ignoring any profits from public sector enterprises and the endowment ω_f, whose sum is likely to be small, $m(s) = (\omega_l - g_l)p_1^*(s)/l_1(s)$. Under the plausible assumption that productivity in sector 1, $1/l_1(s)$, depends only on domestic conditions, we have

$$\pi_i = E_s[p_i^*(s)/m(s)] = E_s\left[\frac{p_i^*(s)}{p_1^*(s)}\right]\cdot E_s[l_1(s)]/(\omega_l - g_l),\ i=1,2,$$

and indeed the use of mean values is valid: $\pi_2/\pi_1 = E_s[p_2^*(s)/p_1^*(s)]$, a result summarised as:

Proposition 3. If, under Assumptions 3 and 4, expected utility is logarithmic ($\rho = 1$), and world prices and domestic productivity are independent variates, then the ratio of the shadow prices of tradable goods is equal to the mean ratio of their respective world prices.

That is to say, the border-price rule is valid without further reference to the possible dependence on states.

The intuition for this result lies in the fact that when $\rho = 1$, the indirect utility function is separable in income and prices: $v = \ln m(s) - \ln \kappa(\mathbf{q}(s))$. This implies, from (13), that the marginal value of public revenue is independent of prices, so that the covariance term vanishes and the expected border price rule indeed applies.[3] The said separability is the reason why the convexity of $1/\kappa(\mathbf{q}(s))$ is exactly balanced by the concavity of ϕ whatever be the degree of substitutability among goods underlying κ. As noted above, κ is absent from π_i $(i = 1, 2)$.

If ρ is not equal to 1, it is seen from (25) that both κ and $m(s)$ will be in play. The argument yielding Proposition 2 then reveals that $\rho = 1$ is a special – and perhaps borderline – case when households are risk averse. Closer inspection of (25) in the light of Propositions 2 and 3 suggests the following conjecture:

$$\frac{\pi_2}{\pi_1} \gtrless E_s \left[\frac{p_2^*(s)}{p_1^*(s)} \right] \quad \text{according as} \quad \rho \gtrless 1. \tag{27}$$

For (25) may be written

$$\pi_i = E_s \left[\frac{p_i^*(s)[\kappa(\mathbf{q}(s))]^{\rho-1}[l_1(s)]^\rho}{[p_1^*(s)]^\rho} \right] \cdot \frac{1}{(\omega_l - g_l)^\rho}, \ i = 1, 2.$$

If $\rho >\ 1$, then the product $[\kappa(\mathbf{q}(s))]^{\rho-1}[l_1(s)]^\rho$ is increasing in both its terms; but if

[3]I am grateful to Francois Bourguignon for this insight. It should be noted that $\rho = 1$ requires the direct utility function to be homogeneous.

$\rho < 1$, it is decreasing in the first, so that their covariance switches sign. In view of Proposition 2 (wherein $\rho = 0$), this proves the conjecture.

Proposition 4. Let good 1 be the numéraire. Then, under Assumptions 3 and 4, the relative shadow price of good 2 is greater or less than its mean relative world price according as ρ is greater or less than 1.

3.2 The size of the errors

It is important to establish how large the error in using the expected values of world prices is likely to be in practice. By continuity, it must be small when ρ is close to 1, so consider the extreme case of risk-neutrality ($\rho = 0$). Recalling that good 1 is the numéraire and (23), we have

$$\frac{\pi_2}{\pi_1} = E_s[p_2^*(s)] + \frac{\mathrm{cov}\,(p_2^*(s), 1/\kappa(\mathbf{q}(s)))}{E_s\,[1/\kappa(\mathbf{q}(s))]}.$$

If the distribution function of the variate P_2^* is strongly concentrated, the covariance term must be small. Suppose, therefore, that the support of the said function is not very narrow, thus allowing the dispersion to be quite large, and exploiting this possibility, let all states be equally probable. Then, defining $h(s) \equiv 1/\kappa(\mathbf{q}(s))$,

$$\frac{\pi_2}{\pi_1} = \frac{E_s[p_2^*(s)h(s)]}{E_s[h(s)]} = \frac{\sum_s p_2^*(s)h(s)}{\sum_s h(s)}.$$

With good 1 as numéraire, $1/p_2^*(s)$ is the barter terms of trade in state s, so the normalisation $E_s[p_2^*(s)] = 1$ is permissible; and with all states equally probable, we have $\sum_s p_2^*(s) = n$, where n is the number of states. Denote the deviation from the mean in state s by $\xi(s) \equiv 1 - p_2^*(s)$, so that $\sum_s p_2^*(s)h(s)/\sum_s h(s) = 1 - \sum_s \xi(s)h(s)/\sum_s h(s)$.

Now suppose the distribution function of P_2^* is also symmetric, and without loss of generality, let the states be ordered such that $p_2^*(1) \leq p_2^*(2) \leq \ldots \leq p_2^*(n)$, where $p_2^*(1) < p_2^*(n)$. Then

$$\sum_s \xi(s)h(s) = \xi(1)(h(1)-h(n))+\xi(2)(h(2)-h(n-1))+\ldots\xi(n/2)(h(n/2)-h(n/2+1)).$$

Let $|(h(k) - h(n+1-k))| \equiv \delta \geq |(h(s) - h(n+1-s))|\ \forall s \neq k$. Hence,

$$\sum_s \xi(s)h(s) \leq \delta \sum_{s=1}^{n/2} \xi(s),$$

which will hold as a strict inequality if the term $h(s) - h(n+1-s)$ varies with s, and *a fortiori* if not all of the terms $h(s) - h(n+1-s)$ have the same sign. It follows that the (proportional) in error using the means of world prices, $\pi_2/\pi_1 - 1$, is at most $-\delta \sum\limits_{s=1}^{n/2} \xi(s) / \sum\limits_{s=1}^{n} h(s)$.

Now, $\sum\limits_{s=1}^{n/2} \xi(s) < (1 - p_2^*(1))n/2\ \forall n > 2$ is the sum of the absolute deviations of $p_2^*(s)$ from the mean value of 1 for all states in which $p_2^*(s) < 1$. In practice, the barter terms of trade $p_1^*(s)/p_2^*(s)$ are unlikely to stray outside the interval $[0.75, 1.25]$, which implies $p_2^*(1) = 0.8$. Under the assumption that all states are equally probable, $\sum\limits_{s=1}^{n} h(s)/n$ is the mean value of the inverse of the cost of living index $\kappa(s)$ and δ is the size of the support of its distribution function. Even allowing for a stochastic level of productivity in the production of good 3, it is very unlikely that the ratio of δ to the mean would exceed 0.3 in practice. Hence, not only is the error in question very unlikely to exceed $-0.2 \times 0.3/2 = -0.03$, but also rather likely to fall some way short of it.

The corresponding error in the shadow wage rate is

$$\frac{\pi_l}{\pi_1 \cdot E_s[1/l_1(s)]} - 1 = \frac{E_s[h(s)/l_1(s)]}{E_s[h(s)] \cdot E_s[/l_1(s))]} - 1 = \frac{n \sum_s h(s)/l_1(s)}{\sum_s h(s) \cdot \sum_s 1/l_1(s)} - 1,$$

where the expression on the far right follows from the assumption that the states are equally probable. Since productivity in the export sector, $1/l_1(s)$, will almost surely lie in the interval $[0.75, 1.25]$, it is seen that the above argument concerning the error in the border price rule also holds for the shadow wage rate.

Having established these qualitative results for the case of risk-neutrality, it will be useful to construct some numerical examples. For simplicity, let productivity in sector 3 be non-stochastic, with $l_3(s) = 1\ \forall s$. As noted above, it is plausible that in a small open economy in which output is produced by unassisted labour, productivity in sector 1, $1/l_1(s)$, is statistically independent of the terms of trade, $1/p_2^*(s)$. Let each of these variates take just two values, both with probability one-half and mean values of unity: $P_2^* \in \{0.8, 1.2\},\ L_1 \in \{0.8, 1.2\}$. There are thus four states, each occurring with probability one-quarter:

$$\begin{aligned} s &= 1 : (p_2^* = 0.8, l_1 = 0.8),\ s = 2 : (p_2^* = 1.2, l_1 = 0.8), \\ s &= 3 : (p_2^* = 0.8, l_1 = 1.2),\ s = 4 : (p_2^* = 1.2, l_1 = 1.2). \end{aligned}$$

The associated contingent producer price vectors are

$$\begin{aligned} (\mathbf{p}(1), w(1)) &= (1, 0.8, 1.25, 1.25),\ (\mathbf{p}(2), w(2)) = (1, 1.2, 1.25, 1.25), \\ (\mathbf{p}(3), w(3)) &= (1, 0.8, 5/6, 5/6),\ (\mathbf{p}(4), w(4)) = (1, 1.2, 5/6, 5/6). \end{aligned}$$

Hence, $E_s[\mathbf{p}(s), w(s)] = (1, 1, 25/24, 25/24)$.

Turning to preferences, consider the symmetric, CES family of utility functions

$$u(\mathbf{x}) = [x_1^k + x_2^k + x_3^k]^{(1-\rho)/k},\ \rho \geq 0,\ k \in (-\infty, 1].$$

The associated indirect utility function is

$$v(\mathbf{q}, m) = \left[\frac{m}{(q_1^r + q_2^r + q_3^r)^{1/r}}\right]^{1-\rho},$$

where $r = k/(k-1)$. Assumption 3 is clearly satisfied. We examine two special cases, which involve all three goods being consumed in strictly positive quantities: $k = -1$, whereby the pairwise elasticity of substitution between any pair of goods is $-1/2$; and $k = 0$, the familiar Cobb-Douglas member of the family, whereby the said elasticity is -1. The associated indirect utility functions are, respectively,

$$v(\mathbf{q}, m) = \left[\frac{m}{\left(q_1^{1/2} + q_2^{1/2} + q_3^{1/2}\right)^2}\right]^{1-\rho},$$

and

$$v(\mathbf{q}, m) = \left[\frac{m}{(3 \cdot q_1\, q_2\, q_3)^{1/3}}\right]^{1-\rho}.$$

Inspecting (25) and noting that consumer prices equal producer prices, we compute

$$\kappa_1(s) = \kappa(1, p_2^*(s)/p_1^*(s), l_3(s)/l_1(s)) \text{ and } \kappa_2(s) = \kappa(p_1^*(s)/p_2^*(s), 1, p_1^*(s)l_3(s)/(p_2^*(s)l_1(s))$$

for goods 1 and 2, respectively.

For $k = -1$, we have, for goods 1 and 2, respectively,

$$\begin{aligned}
\kappa_1(1) &= (1^{1/2} + 0.8^{1/2} + 1.25^{1/2})^2 = 9.075,\ \kappa_1(2) = (1^{1/2} + 1.2^{1/2} + 1.25^{1/2})^2 = 10.326, \\
\kappa_1(3) &= (1^{1/2} + 0.8^{1/2} + (5/6)^{1/2})^2 = 7.881,\ \kappa_1(4) = (1^{1/2} + 1.2^{1/2} + (5/6)^{1/2})^2 = 9.050,
\end{aligned}$$

and

$$\kappa_2(1) = (1.25^{1/2} + 1^{1/2} + 1.5625^{1/2})^2 = 11.344,\ \kappa_2(2) = ((5/6)^{1/2} + 1^{1/2} + (25/24)^{1/2})^2 = 8.605,$$
$$\kappa_2(3) = (1.25^{1/2} + 1^{1/2} + (25/24)^{1/2})^2 = 9.851,\ \kappa_2(4) = ((5/6)^{1/2} + 1^{1/2} + (25/24)^{1/2})^2 = 7.542.$$

If households are risk neutral ($\rho = 0$), (25) yields $\pi_1 = E_s[1/\kappa(s)] = 0.1111$ and $\pi_2 = E_s[p_2^*(s)/\kappa(s)] = 0.1096$. Hence,

$$\frac{\pi_2}{\pi_1} = 0.9865 < E_s\left[\frac{p_2^*(s)}{p_1^*(s)}\right] = 1.$$

The border-price rule is invalid, in keeping with Proposition 1; and with good 1 as numéraire, the covariance of the world price of good 2 and the shadow price of public income is negative, in keeping with Proposition 2. The expected value of $\kappa(s)$ is 9.083, so that $1/E_s[\kappa(s)] = 0.1101 < 0.1111 = E_s[1/\kappa(s)]$. At just over 1 per cent, however, the error resulting from the use of mean values is small indeed.

For $k = 0$ and risk-neutral households, similar calculations yield $\pi_1 = E_s[1/\kappa(s)] = 0.6966$ and $\pi_2 = E_s[1/\kappa(s)] = 0.6871$. Hence,

$$\frac{\pi_2}{\pi_1} = 0.9864 < E_s\left[\frac{p_2^*(s)}{p_1^*(s)}\right] = 1,$$

and the error is virtually identical.

These small errors carry over to the shadow wage rate, which is obtained from (24). For $k = -1$,

$$\pi_l = E_s[1/(\kappa(s)l_1(s))] = 0.1141 < \pi_1 \cdot E_s[1/l_1(s)] = 0.1157,$$

so that the covariance term is negative, whereby the choice of the export good as numéraire contributes to keeping the proportional deviation small. By assumption, $l_3(s) = 1\,\forall s$, so that $\pi_3 = \pi_l = 0.1141$, and the same applies to the shadow price of good 3. For $k = 0$, the absolute deviation is a little larger:

$$\pi_l = E_s[1/(\kappa(s)l_1(s))] = 0.7158 < \pi_1 \cdot E_s[1/l_1(s)] = 0.7256.$$

As noted in Section 3.1, if households are risk averse, the level of income will come into play. In the light of Propositions 3 and 4, we examine the relatively strong risk aversion implied by $\rho = 2$. Recalling (25) and noting that $m(s) = w(s)\omega_l - t(s)$, $w(s) = p_1^*(s)/l_1(s)$ and $t(s) = w(s)g_l$, we have

$$\pi_i = E_s\left[\frac{p_i^*(s)\kappa(\mathbf{q}(s))[l_1(s)]^2}{[p_1^*(s)]^2}\right] \cdot \frac{1}{(\omega_l - g_l)^2},\ i = 1, 2,$$

where $p_1^*(s) = 1\ \forall s$. For $k = -1$, we have $E_s[p_1^*(s)\kappa(\mathbf{q}(s)) \cdot [l_1(s)]^2] = 9.1995$ and $E_s[p_2^*(s)\kappa(\mathbf{q}(s)) \cdot [l_1(s)]^2] = 9.453$ for goods 1 and 2, respectively. Hence, in keeping with Proposition 4,

$$\frac{\pi_2}{\pi_1} = 1.0276 > \frac{E_s[p_2^*(s)]}{E_s[p_1^*(s)]} = 1,$$

albeit still a weak departure from the ratio of mean values. For $k = 0$, we have, likewise,

$$\frac{\pi_2}{\pi_1} = \frac{1.5083}{1.4676} = 1.0277 > E_s\left[\frac{p_2^*(s)}{p_1^*(s)}\right] = 1.$$

Turning to the shadow wage rate, for $k = -1$, the key term is $E_s[p_1^*(s)\kappa(\mathbf{q}(s)) \cdot l_1(s)]$, wherein $l_1(s)$ now enters only linearly. For $k = 1$, its value is 8.9595 , so that

$$\frac{\pi_l}{E_s[p_1^*(s)\kappa(\mathbf{q}(s)) \cdot [l_1(s)]^2] \cdot E_s[1/l_1(s)]} = \frac{8.9595}{9.5831} = 0.9349.$$

For $k = 0$, the ratio is

$$\frac{\pi_l}{E_s[p_1^*(s)\kappa(\mathbf{q}(s)) \cdot [l_1(s)]^2] \cdot E_s[1/l_1(s)]} = \frac{1.4292}{1.5288} = 0.9348.$$

The errors are now rather large and run in the same direction. This stems from the fact that the productivity parameter $1/l_1(s)$ enters the numerator and denominator in different ways, and so leads to substantial covariance between $\pi_1(s)$ and the said parameter.

4 Commodity Taxes

Under the assumption that the entire endowment of labour is supplied completely inelastically, a proportional tax on all goods at the same rate is equivalent to a lump-sum tax. Let the tradable goods be non-taxable, leaving a tax on the non-tradable, $t_3(s)$, to yield whatever revenue is needed to balance the government's budget. The consumer price of good 3 is then $q_3(s) = p_3(s) + t_3(s)$, and the consumer price vector is

$$\mathbf{q}(s) = (p_1^*(s), p_2^*(s), (p_1^*(s)l_3(s)/l_1(s)) + t_3(s)) .$$

Proceeding as in Section 3.1, a change in ω_f now results in a change in $t_3(s)$ so as to preserve a balanced budget. We have, using a little manipulation,

$$d\omega_f = -d(t_3(s) \cdot x_3(s)) = -x_3(s) \left(1 + \frac{t_3(s)}{q_3(s)} \cdot \epsilon_3(s)\right) dt_3(s),$$

where $\epsilon_3(s)$ is the own price elasticity of the uncompensated demand for good 3 in state s. Recalling that the (contingent) producer price of good 3 is fixed, so that

$\partial q_3(s)/\partial t_3(s) = 1$, the corresponding marginal change in $v^0(s)$ is

$$\frac{\partial v^0(s)}{\partial \omega_f} = \frac{\partial v^0(s)}{\partial q_3(s)} \cdot \frac{\partial t_3(s)}{\partial \omega_f} = -\frac{1}{x_3(s)\left(1 + \frac{t_3(s)}{q_3(s)} \cdot \epsilon_3(s)\right)} \cdot \frac{\partial v^0(s)}{\partial q_3(s)}.$$

Using Roy's identity, we then obtain

$$\frac{\partial v^0(s)}{\partial \omega_f} = \mu(s) = \frac{\partial v^0(s)/\partial m(s)}{\left(1 + \frac{t_3(s)}{q_3(s)} \cdot \epsilon_3(s)\right)} \equiv \gamma_3(s) \cdot \partial v^0(s)/\partial m(s). \tag{28}$$

It is seen that, unless the demand for good 3 is completely price-inelastic, the value of $\gamma_3(s)$ is greater than unity, so that the gain in welfare from an additional unit of the endowment ω_f – perhaps in the form of a gift – exceeds the value of the marginal utility of private income in the state in question. This is a simple consequence of the fact that the government is raising revenue through distortionary taxation.

The term $\gamma_3(s)$ will intrude throughout, so it will pay to examine it more closely. Let ω_f and the profits from public production be zero, so that the expenditure $w(s) \cdot g_l$ must be financed by the revenue $t_3(s) \cdot x_3(s)$. Then

$$\frac{t_3(s)}{q_3(s)} = \frac{g_l}{g_l + l_3(s)x_3(s)},$$

where the r.h.s. is the ratio of employment in the production of the public good to the sum of employment in such production and sector 3 combined. Observe that $x_3(s)$ depends not only on $l_3(s)$ itself, but also on $p_2^*(s)$ and $l_1(s)$, variations in all of which induce a variety of income and substitution effects. In practice, the said ratio is likely to be in the range of 20 to 30 percent, and the value of $\epsilon_3(s)$ should not be very far from -1. Hence, the value of the denominator is likely to fall in the interval [0.5, 0.9], which implies a premium on public income in the interval $[0.1, 1]$. In any event,

$$\gamma_3(s) = 1 + \frac{g_l}{g_l + l_3(s)x_3(s)} \cdot \epsilon_3(s), \tag{29}$$

which will vary with s in a complicated way, even when $\epsilon_3(s)$ is constant (as will hold when preferences are Cobb-Douglas).

Turning to shadow prices, analogously to the derivation of (22), we have

$$\pi_i = E_s \left[\gamma_3(s) \cdot \frac{\partial \phi[m(s)/\kappa(\mathbf{q}(s))]}{\partial m(s)} \cdot p_i^*(s) \right], \ i = 1, 2. \tag{30}$$

If households are risk neutral, this specialises to, recalling that good 1 is the numéraire,

$$\pi_1 = E_s \left[\frac{\gamma_3(s)}{\kappa(\mathbf{q}(s))} \right]$$

and

$$\pi_2 = E_s \left[\gamma_3(s) \cdot \frac{p_2^*(s)}{\kappa(\mathbf{q}(s))} \right] = \pi_1 \cdot E_s[p_2^*] + \operatorname{cov}\left(\gamma_3(s), \frac{p_2^*(s)}{\kappa(\mathbf{q}(s))} \right).$$

In general, the covariance term will not be zero, so that the border-price rule will almost surely not hold. Whether Proposition 2 continues to hold, in general, also seems doubtful, though it will do so if $\epsilon_3(s)$ is sufficiently close to zero.

If households are risk averse, inspection of (25) reveals that the intrusion of $\gamma_3(s)$ will also invalidate the arguments yielding Propositions 3 and 4. For each particular constellation of the distributions of the exogenous variates and preferences, it seems plausible that there will exist a borderline value of ρ analogous to $\rho = 1$ with lump-sum taxation. We now investigate this matter.

4.1 Cobb-Douglas preferences

The attraction of Cobb-Douglas preferences is that the equilibrium values of all variables can be derived in closed form. We have

$$v(\mathbf{q}(s)) = (1-\rho)^{-1}\left(\frac{m(s)}{q_1(s)^{1-\alpha_2-\alpha_3}q_2(s)^{\alpha_2}q_3(s)^{\alpha_3}}\right)^{1-\rho}. \tag{31}$$

Choosing good 1 as numéraire, let taxes be levied on one or the other of goods 2 and 3. Let ω_f and the profits from public production be zero, so that in order to finance g_l, the tax on good i must satisfy

$$t_i(s)x_i(s) = [q_i(s) - p_i(s)]x_i(s) = w(s)g_l, \quad i = 2, 3.$$

Normalising the endowment ω_l to unity, we have $q_i(s)x_i(s) = \alpha_i w(s)$. Hence, $q_i(s) = \alpha_i p_i(s)/(\alpha_i - g_l)$, $i = 2, 3$, where $g_l < \max(\alpha_2, \alpha_3)$ if g_l is to be feasible.

Suppose only the non-tradable is taxed. Then, recalling (5)-(7), noting that $w(s) = p_1^*(s)/l_1(s) = 1/l_1(s)$ and assuming that $l_3(s) = 1\ \forall s$, we obtain

$$y_1(s) = (\alpha_1 + \alpha_2)/l_1(s),\ y_2(s) = 0,\ y_3(s) = x_3(s) = \alpha_3 - g_l.$$

Strikingly, production and consumption of the non-tradable are independent of s. The same holds for $\gamma_3(s)$: from (29), we have $\gamma_3(s) = (\alpha_3 - g_l)/\alpha_3$.

Now let the economy receive a gift of ω_f. This can be used to finance part of $w(s)g_l$, thereby permitting a reduction in $t_3(s) = g_l/[(\alpha_3 - g_l)l_1(s)]$. The change in $q_3(s)$ so induced by a marginal change in ω_f is

$$\frac{\partial q_3(s)}{\partial \omega_f} = \frac{\alpha_3}{(\alpha_3 - g_l)^2 l_1(s)} \cdot \frac{\partial g_l}{\partial \omega_f} = \frac{\alpha_3}{(\alpha_3 - g_l)^2 l_1(s)} \cdot \frac{-1}{w(s)} = \frac{-\alpha_3}{(\alpha_3 - g_l)^2},$$

which is independent of s.

In view of Proposition 3, the case of log-expected utility, $v(s) = \ln[m(s)/\kappa(\mathbf{q}(s)]$, is of particular interest, as a possible borderline case. We have

$$\mu(s) = \frac{\partial v^0(s)}{\partial \omega_f} = \frac{\partial v^0(s)}{\partial q_3(s)} \cdot \frac{\partial q_3(s)}{\partial \omega_f} = \frac{\alpha_3 l_1(s)}{\alpha_3 - g_l}\,.$$

Hence, $\pi_1 = E_s[p_1^*(s)\mu(s)] = \alpha_3 E_s[l_1(s)]/(\alpha_3 - g_l) = E_s[p_2^*(s)\mu(s)] = \pi_2$ in virtue of the assumption that $p_2^*(s)$ is independent of $l_1(s)$. A similar argument establishes that the same holds when good 2 is taxed instead, and hence also for any combination of taxes on goods 2 and 3. We have therefore extended Proposition 3 to cover commodity taxes.

Proposition 5. If, under Assumptions 3 and 4, expected utility is logarithmic ($\rho = 1$), preferences over goods are Cobb-Douglas, world prices and domestic productivity are independent variates, and only commodity taxes are available, then the ratio of the shadow prices of tradable goods is equal to the mean ratio of their respective world prices.

This finding invites the question of whether Proposition 4 also holds under such a tax restriction when preferences are Cobb-Douglas. If there is risk-neutrality, then

$$\mu(s) = \frac{w(s)}{\kappa(\mathbf{q}(s))} \cdot \frac{\alpha_3 l_1(s)}{\alpha_3 - g_l} = \frac{1}{\kappa(\mathbf{q}(s))} \cdot \frac{\alpha_3}{\alpha_3 - g_l}\,.$$

Recalling (31), we have

$$\pi_1 = \left(\frac{\alpha_3 - g_l}{\alpha_3}\right)^{\alpha_3} \cdot E_s\left[p_2^*(s)^{-\alpha_2} l_1(s)^{\alpha_3}\right] \text{ and } \pi_1 = \left(\frac{\alpha_3 - g_l}{\alpha_3}\right)^{\alpha_3} \cdot E_s\left[p_2^*(s)^{1-\alpha_2} l_1(s)^{\alpha_3}\right],$$

so that

$$\frac{\pi_2}{\pi_1} = \frac{E_s[p_2^*(s)^{1-\alpha_2}]}{E_s[p_2^*(s)^{-\alpha_2}]} < 1$$

in virtue of Jensen's inequality and the fact that the functions $p_2^*(s)^{1-\alpha_2}$ and $p_2^*(s)^{-\alpha_2}$ are, respectively, strictly concave and strictly convex in $p_2^*(s)$.

If $\rho = 2$, similar calculations yield, in contrast,

$$\frac{\pi_2}{\pi_1} > 1,$$

also in keeping with Proposition 4. It turns out, however, that both deviations from unity are likely to be small in practice. Given the values of $p_2^*(s)$ and $\alpha_3 = 1/3$ in Section 3.2, for example, $\pi_2/\pi_1 = 0.9865$ when $\rho = 0$ and 1.0069 when $\rho = 2$. With such deviations, the border-price rule using mean world prices involves negligible errors for all plausible values of $\rho \neq 1$.

5 Project Uncertainty

Individual projects are undertaken in the uncertain environment described by the joint distribution of the system's variates, exogenous and endogenous alike; but that is not the end of the matter where a particular project's social profitability is concerned. For it is rare indeed that a project works exactly as laid out in the plan drawn up by its designers, engineers and managers.

According to the standard definition, a public sector project is a change in that sector's net supply vector. In an uncertain environment, any such change may depend, *ex post*, on the realised state s, a complication that demands some discussion. Expressed formally, a project in the planning stage is the *ex ante* n-vector $(\Delta\mathbf{z}(1), \ldots, \Delta\mathbf{z}(n))$: there is a project for each and every state. If the project always functions as planned, contingent on s, its expected social profitability will be $E_s[\boldsymbol{\pi}(s) \cdot \Delta\mathbf{z}(s)]$; for the contingent shadow price vectors $\boldsymbol{\pi}(s)$ measure the changes in welfare that result from an extra unit of supply of each good in the state in question. In the very special case where the project is expected to function exactly as planned and in the same way regardless of the realised state, the expected social profit will be $E_s[\boldsymbol{\pi}(s)] \cdot \Delta\mathbf{z} = \boldsymbol{\pi} \cdot \Delta\mathbf{z}$. The *ex ante* shadow prices, which relate to delivery whatever state s is realised, should be used; for although, by hypothesis, $\Delta\mathbf{z}$ is non-stochastic, the social value of this outcome must reflect the fact that it will occur in a stochastic environment.

In practice, things are almost certainly less tidy where $\Delta\mathbf{z}(s)$ is concerned. Building a road may run into unforeseen difficulties with drainage; new crop varieties may not respond to fertilisers on farmers' fields as they do on experimental stations; a steel plant may not get along well with local coal; or, on a happier note, school meals may yield unexpectedly large improvements in children's physical and cognitive development. What all of these examples have in common is that they are outcomes that are arguably independent of the environmental state s: that is to say, despite the best-laid plans, $\Delta\mathbf{z}(s)$ is itself stochastic. For each $s \in S$, let the variate $\Delta Z_i(s)$ denote the change in the net supply of good i. If the $Z_i(s)$ are distributed independently of s, the expected social profit of the project planned as $(\Delta\mathbf{z}(1), \ldots, \Delta\mathbf{z}(n))$ is $E_s[\boldsymbol{\pi}(s) \cdot \Delta\mathbf{z}(s)] = \boldsymbol{\pi} \cdot E_s[\Delta\mathbf{z}(s)]$, so that using the mean value of $\Delta\mathbf{z}(s)$ is valid. If, however, the $\Delta Z_i(s)$ are not distributed independently of s, then neglecting the covariance terms may lead to significant errors.

Those charged with assessing a project's social profitability in practice will surely regard the need to specify $(\Delta\mathbf{z}(1), \ldots, \Delta\mathbf{z}(n))$ as a counsel of perfection; for getting the engineers, specialists and managers to agree on *any* definite $\Delta\mathbf{z}$ may often seem hard enough. Confronted with the complications just discussed, what, then, are practitioners to do? One tempting approach is to insist that the engineers and managers provide estimates of the mean values of the $\Delta Z_i(s)$, ideally with their ranges, but without mentioning s. Thus armed, the practitioner can calculate $\boldsymbol{\pi} \cdot E_s[\Delta\mathbf{Z}(s)]$ and then perform sensitivity analysis using what seem to be plausible values of the covariance terms arising from any suspected dependence of the $\Delta Z_i(s)$ on s.

5.1 An example

To illustrate what is involved in more detail, consider the following project, which is to be undertaken in the setting described in Section 3.2:

$$\Delta\mathbf{z}(1) = \Delta\mathbf{z}(2) = (4, -1, -1, -1);\ \Delta\mathbf{z}(3) = \Delta\mathbf{z}(4) = (2.5, -1, -1, -1).$$

The project may be thought of as an innovation in the export sector that involves inputs of the imported good and the non-tradable, as well as labour. Input levels are independent of s, but output varies in response to purely domestic shocks, which affect productivity in existing activity in that sector, as represented by fluctuations in the parameter l_1. For the first example ($k = -1$), the vectors of contingent shadow prices under risk-neutrality ($\rho = 0$) are

$$(\boldsymbol{\pi}(1), \pi_l(1)) = (0.1102, 0.0882, 0.1378, 0.1378);$$

$$(\boldsymbol{\pi}(2), \pi_l(2)) = (0.0968, 0.1162, 0.1210, 0.1210);$$

$$(\boldsymbol{\pi}(3), \pi_l(3)) = (0.1269, 0.1015, 0.1058, 0.1058);$$

$$(\boldsymbol{\pi}(4), \pi_l(4)) = (0.1105, 0.1326, 0.0921, 0.0921).$$

The corresponding vectors when $\rho = 2$ are

$$\begin{aligned}(\boldsymbol{\pi}(1), \pi_l(1)) &= (5.808, 7.260, 7.260, 7.260);\\ (\boldsymbol{\pi}(2), \pi_l(2)) &= (6.609, 5.507, 8.261, 8.261);\\ (\boldsymbol{\pi}(3), \pi_l(3)) &= (11.349, 14.185, 9.457, 9.457);\\ (\boldsymbol{\pi}(4), \pi_l(4)) &= (13.032, 10.860, 10.860, 10.860).\end{aligned}$$

where the common scalar $1/(\omega_l - g_l)^2$ may be normalised to 1.

As described above, the net output vector is known in each state s: that is to say, the project is confidently expected to function as planned, conditional on s. We therefore compute $E_s[\boldsymbol{\pi}(s) \cdot \Delta\mathbf{z}(s)]$ in order to evaluate its social profitability. When $\rho = 0$, $E_s[\boldsymbol{\pi}(s) \cdot \Delta\mathbf{z}(s)] = 0.0612$; and when $\rho = 2$, $E_s[\boldsymbol{\pi}(s) \cdot \Delta\mathbf{z}(s)] = 0.283$ is likewise positive, whereby the difference in their absolute magnitudes has no significance. The project is socially profitable under both preferences for risk-bearing. In this connection, it should be remarked that its input vector is independent of s, so that the expected social profit can be equally well computed as $E_s[\pi_1(s)\Delta z_1(s)] - \pi_2\Delta z_2 - \pi_3\Delta z_2 - \pi_l\Delta z_l$.

There remains the important possibility that things do not necessarily turn out as planned, conditional on s. Suppose, for example, that in the system-wide states wherein productivity in sector 1 is low ($s = 3, 4$), output is not certain to be 2.5, but could also take the still lower value 2, say with probability 0.5. Since the project is sufficiently small, this additional source of risk will have no effect on shadow prices; so that in the foregoing calculations, one simply substitutes the mean value of output in states 3 and 4, namely 2.25, for the given contingent value of 2.5. This yields $E_s[\boldsymbol{\pi}(s) \cdot \Delta\mathbf{z}(s)] = 0.0026$ when $\rho = 0$ and $E_s[\boldsymbol{\pi}(s) \cdot \Delta\mathbf{z}(s)] = -1.241$ when $\rho = 2$. What can be termed the project's 'idiosyncratic' risk makes it unprofitable when risk aversion is sufficiently strong.

5.2 An alternative: the distribution function of net returns

It might be argued that decision-makers are unlikely to find shadow prices, as signals of social scarcity, especially transparent objects. Yet they may well be interested not only in a project's net returns, but also in the dispersion thereof, with particular reference to the downside risks. If the states s can be sufficiently parsimoniously described, then given the n-vector $(\Delta\mathbf{z}(1), \ldots, \Delta\mathbf{z}(n))$, deriving the cumulative distribution function of the project's net returns will be a practical possibility, using Monte Carlo methods. With the project thus described – and presented – as a lottery, there appears to be a basis for the decision of whether to accept it.

The public sector trades at producer prices and so would realise (net) profits in the amount of $((\mathbf{p}(s), w(s)) \cdot \Delta\mathbf{z}(s))$ in state s. Suppose only lump-sum taxes are employed, so that this amount is distributed to households as a change in t. Now although consumer prices are then equal to producer prices, the latter depend on s, so that the change in *real* net income is the (net) profit deflated by the price index in the state in question, $\kappa(s)$. Associated with each state is the probability of its occurrence, which completes the contribution of the state to the whole distribution function of the project's real net returns.

To illustrate, consider once more the project analysed in Section 5.1. For the first example ($k = -1$), we have the following state-contingent net returns:

$$
\begin{aligned}
(\mathbf{p}(1), w(1)) \cdot \Delta\mathbf{z}(1) &= 0.7,\ \kappa(1) = 9.075,\ \Delta m(1)/\kappa(1) = 0.7/9.075 = 0.0771; \\
(\mathbf{p}(2), w(2)) \cdot \Delta\mathbf{z}(2) &= 0.3,\ \kappa(2) = 10.326,\ \Delta m(2)/\kappa(2) = 0.3/10.326 = 0.0291; \\
(\mathbf{p}(3), w(3)) \cdot \Delta\mathbf{z}(3) &= -0.217,\ \kappa(3) = 7.881,\ \Delta m(3)/\kappa(3) = -0.217/7.881 = -0.0275; \\
(\mathbf{p}(4), w(4)) \cdot \Delta\mathbf{z}(4) &= -0.617,\ \kappa(4) = 9.050,\ \Delta m(4)/\kappa(4) = -0.617/9.050 = -0.0682.
\end{aligned}
$$

Since each state occurs with probability 0.25, the expected value of the project's net real returns is $(0.0771 + 0.291 - 0.275 - 0.0682)/4 = 0.002625$. On that criterion, therefore, the project should be undertaken. Here, it should be remarked that the resulting small number does not necessarily imply that the project is marginal in some sense; for its magnitude can be increased by renormalising κ or a simple scaling up the project's size.

The distribution is strongly bimodal, with a heavy left-hand tail, which may well attract a decision-maker's attention. Is that warranted? The drawback of the whole approach is that the lottery generated by the project should be evaluated in relation to the much larger lottery of national income, so that the question of whether their respective outcomes are correlated arises once more. For the above project, they are positively so by assumption. With full employment, real national income is proportional to the wage rate deflated by κ, the vector of whose values is $(0.138, 0.121, 0.106, 0.092)$. It is seen that both national income and the project's net returns are lowest in states 3 and 4. Hence, although the absolute magnitude of the net return in state 1 exceeds that in state 4, and so ensures a positive expected value of net returns, the larger pay-off in the former state could be outweighed in value by the loss in the latter if risk aversion is strong enough. The fundamental advantage of the procedure using shadow prices is that all this is taken into account when deriving them. It is established in Section 5.1

that the project is indeed socially profitable when $\rho = 2$; but that cannot be asserted by inspecting the distribution function.

The conclusion to be drawn from all this is that if the distribution function of a project's net returns is to be used as the basis for making decisions, due caution is in order. There are many decision-makers in the public sector, and left to themselves, each may not evaluate the lotteries that land on his or her desk in the same way. The use of a common set of shadow prices imposes uniformity on the process of evaluation.

6 Concluding Discussion

The rule that, under carefully specified conditions, the use of expected values to establish whether a project will improve welfare is intellectually satisfying and practically attractive. For a small open economy, this entails, in particular, estimating the average world prices of tradable goods. The following complication arises in connection with placing a social value on such goods. If, for example, an extra unit of an exportable becomes available when its world price turns out to be high, then, *cet. par.*, income will tend to be high and hence the social value of the said unit will tend to be low: the exogenous stochastic world price and the social value of the available unit of the good will covary. Since world prices also determine the domestic prices of factors and nontradables, these, too, are stochastic, and the sources of covariance between events and valuations become yet more numerous. These effects ramify throughout the whole system in such a way as to make the use of average world prices and average levels of domestic productivity in various branches of domestic production invalid in most circumstances. They do not arise when there is a single commodity, though there can be other reasons why a project's net returns may be correlated with national income in a single-commodity world.

The ensuing complications for the estimation of shadow prices of goods and labour are rather daunting, so it is natural to ask under what circumstances the errors involved in employing expected values will be small, at least in relation to all the other uncertainties that shroud an investment undertaking. The following result provides such a basis. If the coefficient of relative risk aversion is 1, preferences over goods can be represented as a function of real income, and public expenditures are financed by lump-sum taxes, then no error is involved in using means. To put it slightly loosely, if preferences over lotteries can be represented by something close to logarithmic expected utility, then it will be safe to use expected values in order to derive shadow prices and estimate the social profitability of projects. Both general arguments and numerical examples indicate that the same holds for a wide range of values of the coefficient of relative risk aversion, including risk-neutrality, with the caveat that the resulting error in the shadow wage rate may be substantial when risk aversion is strong. If the government can impose only (distortionary) commodity taxes and preferences over goods are Cobb-Douglas, the error involved in using means will also be small for all values of the coefficient of relative risk aversion other than 1. Since the latter restriction on preferences over goods is strong, whereas tax regimes are invariably distortionary in practice, there is, as is customary to conclude, scope for further research.

Appendix

Proof of Lemma 1. Since g is a concave function,

$$g(\alpha\mathbf{x} + (1-\alpha)\mathbf{x}') \geq \alpha g(\mathbf{x}) + (1-\alpha)g(\mathbf{x}') \ \forall\alpha \in [0,1].$$

Define $f(\mathbf{x}) = 1/g(\mathbf{x})$. Then a little manipulation yields

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{x}') \leq \frac{f(\mathbf{x}) \cdot f(\mathbf{x}')}{\alpha f(\mathbf{x}') + (1-\alpha)f(\mathbf{x})} \ \forall\alpha \in [0,1].$$

Now,

$$\frac{f(\mathbf{x}) \cdot f(\mathbf{x}')}{\alpha f(\mathbf{x}') + (1-\alpha)f(\mathbf{x})} \leq \alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{x}') \ \forall\alpha \in [0,1].$$

For by cross-multiplying and simplifying, it is seen that this condition holds if and only if $[f(\mathbf{x}) - f(\mathbf{x}')]^2 \geq 0$, with equality if and only if $f(\mathbf{x}) = f(\mathbf{x}')$. Hence,

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{x}') \leq \alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{x}') \ \forall\alpha \in [0,1],$$

which holds as a strict inequality $\forall\alpha \in (0,1)$ and $\forall(\mathbf{x},\mathbf{x}')$ s.t. $f(\mathbf{x}) \neq f(\mathbf{x}')$.

If $\mathbf{x}, \mathbf{x}'$ are s.t. $f(\mathbf{x}) = f(\mathbf{x}')$, then $f(\mathbf{x}) = f(\mathbf{x}') \geq f(\alpha\mathbf{x} + (1-\alpha)\mathbf{x}')$, so that

$$f(\alpha\mathbf{x} + (1-\alpha)\mathbf{x}') \leq \alpha f(\mathbf{x}) + (1-\alpha)f(\mathbf{x}') \ \forall\alpha \in [0,1],$$

which holds as a strict inequality $\forall\alpha \in (0,1)$, unless f is affine. If f is affine, however, $g(\mathbf{x})$ will be strictly convex, which contradicts the assumption that g is concave. Q.E.D.

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